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Research & Development Report

# Impulse Response Operators for Structural Complexes


by

G. Maidanik

J. Dickey


DTRC-90/011 Impulse Response Operators for Structural Complexes

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A number of sequential and parallel procedures for analyzing the response of structural complexes, subjected to various drives, are briefly developed and discussed. The impulse response vector operator is defined in terms of impulse response operators, each associated with a unique path between the localized position of a test drive and a localized position of observation. The drive is, correspondingly, a vector and the response is the scalar product of the impulse response operator and the drive vectors. A sequential procedure of subdividing a structural complex into a number of coupled dynamic systems is stated. The formalism is then stated in terms of matrices and vectors; e.g., the response is a vector; each element represents the response of a specific dynamic system, the impulse response operator is a matrix; the off-diagonal elements describe the couplings between the dynamic systems, etc. If the dynamic systems are chosen so that each, in isolation, can be described in terms of an eigen-impedance operator, then, in addition, a modal analysis can be applied to the multiple dynamic systems that compose the model of the structural complex. In the modal analysis, however, the ranks of the impulse response matrix, the response vector, and the drive vector, are swollen by the modal count, usually rendering the matrix equation for the response unwieldy. The modal approach may be substituted by a wave approach. In this parallel approach, the propagations in the dynamic systems are described by impulse response operators that are commensurate with those pertaining to boundlessly extrapolated dynamic systems. The finiteness of the dynamic systems are accounted for by junction matrices; a junction defines the boundaries through which dynamic systems interact either with each other (transmissions) or with self (reflections). As in the modal approach, in this wave approach, the resulting formalism is, again, rather unwieldy. It is shown that considerable reductions and simplifications are attained if the complex can be modeled by spatially one-dimensional dynamic systems.

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# INTRODUCTION

The analysis of the response of a driven complex structure is relevant not only to dealing with acoustical problems, but also to dealing, among others, with optical, biological, and economical problems. This generalization and encompassment of the solutions to the response of structural complexes may have encouraged wider interest and contributed to the variety of approaches to the analyses of these generic problems. In this paper the acoustical problems are implicitly epitomized.

The analysis of the response of a (structural) complex to various external drives is often well nigh impossible. One must then resort to some reduction in the definitions and descriptions of the complex and the external drive system to render the analysis manageable. A reduction is usually achieved by modeling the complex and the external drive. The model is chosen so that the analysis can be performed. Some modeling insists on details so that one-to-one correspondence between the response of the model and the actual complex is substantially preserved. Measures and devices (active and/or passive) that are instituted to control the response (or responses) may be compatibly modeled. The elaborated model may then be exercised and the results substantially related to the controlled complex. In such applications one may estimate the benefits of the instituted controls also on the basis of one-to-one correspondence. On the other hand, some modeling may insist merely on a phenomenological correspondence ; one may be satisfied then merely with descriptions of the response that manifest a phenomenon (or phenomena) that is common to the simplified model and the actual (structural) complex. Measures and devices (passive and/or active) that may be instituted to control the phenomenon (or phenomena) may then be examined on the elaborated model of the complex and phenomenologically related, by analogy, to the controlled complex. In such applications one may not, however, demand precise estimate of the benefits accrued from the institution of the controls; beneficial trends and inclinations is all that can be expected. Moreover, not all phenomenon (or phenomena) can be accounted for in simplified models and, therefore, caution should be exercised in any phenomenological correspondence procedures. In this paper the complex is variously modeled in the milieu of

phenomenological correspondence. Moreover, it is not modeled with the intent of establishing a particular description, but rather with the intent of exploring analytical techniques and procedures that may be employed in deriving general phenomenological descriptions. Although this aim renders the paper somewhat abstract, it is hoped that, nonetheless, it will be found useful by some readers. Finally, the relationship is established between modeling that utilizes a multiple spatial dimensionality of the actual complexes and modeling that relies solely on a single spatial dimensionality. This is of significance to the modeling used in recent papers [1-5].

The basic description of the complex is cast in terms of an impulse response function $g(\hat{x}|\hat{x}', t|t')$ so that the external drive $p_e(\hat{x}', t')$, defined at the position $\{\hat{x}', t'\}$, will generate a response $p(\hat{x}, t)$ at the observation position $\{\hat{x}, t\}$; i.e.,

$$\int g(\hat{x}|\hat{x}', t|t')\, d\hat{x}'dt'\, p_e(\hat{x}',t') = p(\hat{x}, t)\ ; \quad \hat{x} = \sum_{s=1}^{s_0} \hat{s}\, x_s\ ; \quad d\hat{x}' = \prod_{s=1}^{s_0} dx_s' \ , \tag{1a}$$

where $\hat{s}$ is a unit vector lying in the spatial coordinate $x_s$ and $[\hat{s}\,\hat{n}] = [\hat{n}\,\hat{s}] = \delta_{sn}$.[1] The complex is three-dimensional when $s_0 = 3$, two-dimensional when $s_0 = 2$, and one-dimensional when $s_0 = 1$. The impulse response function is pure if it is a functional only of the parameters and quantities that describe the complex; it is independent of the drive $p_e(\hat{x}', t')$ and the response $p(\hat{x}, t)$. Situations may arise in which it may be convenient and instructive to express equation (1a) in spaces other than $\{\hat{x}, t\}$; e.g., in the $\{\hat{x}, \omega\}$, $\{\hat{k}, t\}$, and $\{\hat{k}, \omega\}$ - spaces, where the wavevector $\hat{k}$ and the frequency $\omega$ are the Fourier conjugates of the spatial vector variable $\hat{x}$ and the

[1]*The caret is used herein to indicate a vectorial representation. A caret over an index indicates a "unit vector"; e.g.,* $\hat{s}$ *in equation (1a). A caret over a variable, e.g.,* $\hat{x}$ *in equation (1a), indicates that it may be multi-dimensional; each component is associated with a given normal direction. A caret over a quantity, e.g.,* $\widehat{g}$ *in equation (8), indicates that it may be multi-path; each component is associated with a given normal path. A "unit path vector" is indicated by a caret over the index of the path; e.g.* $\widehat{v}$ *in equation (8).*

temporal variable t, respectively. The spaces of convenience may be chosen on the basis of the external drive, the response, or both. For example, equation (1a) may be expressed in the forms

$$\int \tilde{g}'(\hat{x}|\hat{x}', t|\omega')\, d\hat{x}'\, d\omega'\, \tilde{p}_e(\hat{x}',\omega') = p(\hat{x}, t) \quad , \tag{1b}$$

$$\int \tilde{g}''(\hat{x}|\hat{x}', \omega|t')\, d\hat{x}'\, dt'\, p_e(\hat{x}',t') = \tilde{p}(\hat{x}, \omega) \quad , \tag{1c}$$

$$\int \tilde{g}(\hat{x}|\hat{x}', \omega|\omega')\, d\hat{x}'\, d\omega'\, \tilde{p}_e(\hat{x}',\omega') = \tilde{p}(\hat{x}, \omega) \quad , \tag{1d}$$

$$\int \tilde{\tilde{g}}'(\hat{x}|\hat{k}', t|t')\, d\hat{k}'\, dt'\, \tilde{\tilde{g}}_e(\hat{k}',t') = p(\hat{x}, t) \quad , \tag{1e}$$

$$\int \tilde{\tilde{g}}''(\hat{k}|\hat{x}', t|t')\, d\hat{x}'\, dt'\, p_e(\hat{x}',t') = \tilde{\tilde{p}}(\hat{k}, t) \quad , \tag{1f}$$

$$\int \tilde{\tilde{g}}(\hat{k}|\hat{k}', t|t')\, d\hat{k}'\, dt'\, \tilde{\tilde{p}}_e(\hat{k}', t') = \tilde{\tilde{p}}(\hat{k}, t) \; ; \quad \hat{k} = \sum_{s}^{s_0} \hat{s}\, k_s \; ; \quad d\hat{k}' = \prod_{s}^{s_0} dk_s' \quad , \tag{1g}$$

$$\int G(\hat{k}|\hat{k}', \omega|\omega')\, d\hat{k}'\, d\omega'\, P_e(\hat{k}', \omega') = P(\hat{k}, \omega) \quad , \tag{1h}$$

and so forth, where typically

$$\tilde{p}(\hat{x}, \omega) = (2\pi)^{-1/2} \int dt\, p(\hat{x}, t) \exp(-i\omega t) \quad , \tag{2a}$$

$$\tilde{\tilde{p}}(\hat{k}, t) = (2\pi)^{-(s_0/2)} \int d\hat{x}\, p(\hat{x}, t) \exp(i\hat{k}\hat{x}) \quad , \tag{2b}$$

$$\tilde{g}(\hat{x}\,|\,\hat{x}', \omega|\omega') = (2\pi)^{-1} \int dt \int dt' \exp(-i\omega t)\, g(\hat{x}\,|\,\hat{x}', t\,|\,t') \exp(i\omega' t') \quad . \tag{2c}$$

One may choose equation (1b) if the external drive happens to be highly localized in the frequency

domain (steady state drive) and yet the response is desired in the temporal domain. On the other hand, if the external drive happens to be highly localized in the temporal domain (transient drive) and one desires the response in the frequency domain one may choose equation (1c). Further, if the external drive is highly localized in the frequency domain and the response is desired also in this domain, equation (1d) may be chosen. Other combinations, involving also the spatial and wavenumber domains can be similarly classified. However, it may transpire that the form of the impulse response function may dictate in part which of the spaces are to be conveniently (and/or expeditiously, within a Fourier transformation,) chosen. For example, situations may arise in which the impulse response function may be stationary in one and/or the other dependent variable; e.g.,

$$g(\hat{x}\,|\,\hat{x}',\ t\,|\,t') = (2\pi)^{-1/2}\, g(\hat{x}\,|\,\hat{x}',\ t-t') \quad , \tag{3a}$$

$$g(\hat{x}\,|\,\hat{x}',\ t\,|\,t') = (2\pi)^{-(s_0/2)}\, g(\hat{x}-\hat{x}',\ t\,|\,t') \quad . \tag{3b}$$

In such situations one finds that

$$\tilde{g}(\hat{x}\,|\,\hat{x}',\ \omega\,|\,\omega') = \tilde{g}(\hat{x}\,|\,\hat{x}',\ \omega)\ \delta(\omega-\omega') \quad , \tag{3c}$$

$$\begin{gathered} \tilde{\tilde{g}}(\hat{k}\,|\,\hat{k}',\ t\,|\,t') = \tilde{\tilde{g}}(\hat{k},\ t\,|\,t')\ \delta(\hat{k}-\hat{k}')\,; \\ \delta(\hat{k}-\hat{k}') = \prod_{s}^{s_0} \delta(\hat{k}_s-\hat{k}'_s) \quad , \end{gathered} \tag{3d}$$

respectively, where typically

$$\tilde{g}(\hat{x}\,|\,\hat{x}',\ \omega) = (2\pi)^{-1/2} \int d\tau\ \tilde{g}(\hat{x}\,|\,\hat{x}',\ \tau)\, \exp(-i\omega\tau) \quad . \tag{3e}$$

From equations (1) and (3) it is apparent that stationarity results in substantial simplifications. It reduces, in the Fourier transform domain that corresponds to that in which the stationarity is

present, an integral operator to an algebraic factor. The availability of such a reduction may dictate the choice of the space of convenience just discussed. Finally, it is convenient, for the most part, to deal with the impulse response integral operator rather than with the impulse response function. The impulse response operator $h(\hat{x}\,|\,\hat{x}', t\,|\,t')$ is related to the impulse response function $g(\hat{x}\,|\,\hat{x}', t\,|\,t')$ in the form

$$h(\hat{x}\,|\,\hat{x}', t\,|\,t') \equiv \int g(\hat{x}\,|\,\hat{x}', t\,|\,t')\, d\hat{x}'\, dt' \cdots \quad ; \tag{4a}$$

e.g.,

$$h(\hat{x}\,|\,\hat{x}', t\,|\,t')\, p_e(\hat{x}',t') \equiv \int g(\hat{x}\,|\,\hat{x}', t\,|\,t')\, d\hat{x}'\, dt'\, p_e(\hat{x}',t') \quad . \tag{4b}$$

where $p_e(\hat{x}',t')$, in equation (4b), is an arbitrary, but well behaved, function of the dependent vector variable $\{\hat{x}', t'\}$ Invariably, except for a few simple structural complexes, the direct derivation of the impulse response functions or operators are difficult and cumbersome. One must then resort to modeling procedures that will give the derivation a chance. In this paper a number of analytical approaches, designed to derive the proper impulse response functions or operators of complex structures, are considered.

It may be useful, at this early stage, to make a precis of the story of the paper. In this way some idea as to where the paper goes will be related before the multitude of equations and the maze of notations take over. The response of a (structural) complex that is excited by an external drive is formulated in terms of an impulse response function and/or operator. The operator is an integral operator that spans the spatial and the temporal extents of the complex and the external drive. This formalism is depicted in equations (1) through (4). As just stated, rarely is the complex and the drive sufficiently simple that they can be reasonably represented in single forms. It is, therefore, proposed to model the complex and the drive so that the description can be decomposed. The decomposition is in reference to unique and significant paths yielding components through which

the response can be estimated by a superposition. The hope is that the analysis in terms of these paths may lead to a set of simpler impulse response operators. A vectorial arrangement of the set, with a corresponding set of drives, may render the formalism simpler in certain applications. A more drastic step in the analysis is then taken. The complex is modeled in terms of elemental constituents -- dynamic systems, each of which is amenable to a reasonable description. The premise is that the impulse response operator associated with each dynamic system may be reasonably derived. The drive is similarly modeled in terms of elemental constituents, each assigned to a dynamic system. The formalism, as expected, is cast in matrix form: The complex is described in terms of an impulse response matrix operator, the external drive in terms of a compatible vector, and the response is then a vector; each element, in the response vector, is the response of a specific dynamic system. A major difficulty in this analytical modeling is the multiplicity of descriptions required and the requirement for the description of the interactions (couplings) among the dynamic systems. Nonetheless, this modeling technique is central to the paper. This modeling technique is of particular appeal in those situations in which the generic and phenomenological properties of the response of a collection of dynamic systems that interact may be more significant than the precise evaluation of that response. Nonetheless, to this point in the development of the analysis, the modeling of the complex in terms of coupled dynamic systems is formally stated and the resulting equation for the response is thus of limited practical utility. The impotence of the equation is evidenced, for example, by its insensitivity to the spatial dimensionality of the dynamic systems. To infuse some practicality into the analysis just developed, two major approaches are taken: 1. the modal approach, 2. the wave approach. There are situations in which the modal and wave approaches can be readily related. The Sommerfield Watson transformation constitute such a relationship [6,7]. However, in this paper, the relationship is not a paramount topic. Rather, in the paper, the modal and the wave analytical procedures are considered as parallel, but distinct, approaches to the analysis of the response of models that cast the essential characteristics of the complex in terms of coupled dynamic systems, as was just formally recounted.

1. In the modal approach the couplings are expressed in terms of coupling impedance operators that are compatible with the impedance operators of the individual isolated dynamic systems. The latter impedance operators are chosen to be eigen-operators so that each defines a set of eigenfunctions and a corresponding set of eigen-values. The sets of eigenfunctions enable one to suppress the spatial dependence of the matrix equation for the response vector, thereby rendering this an equation of temporal dependence only. The spatial dependence of the response can be recovered if the modal matrix equation for the temporal portion of the response is resolved and the eigenfunctions are explicitly known. It emerges, however, that the modal matrix equation for the temporal portion of the response is of higher rank, by the modal count, than the original equation. It appears, therefore, that the modal approach is useful when the modal count is low and the eigenfunctions are known. Usually, however, the increase in rank and the general difficulty of specifying the eigenfunctions for most dynamic systems, makes the modal approach impractical, unless statistical measures are brought to bear.

2. In the wave approach the model is defined in terms of extrapolated matrix propagators, terminal position vectors, and junction matrices. A propagator, for a given dynamic system, is defined in terms of propagation in which back-scatterings, from regions outside the insitu boundaries, are eliminated by an appropriate spatial extrapolations of the dynamic system. The terminal position vectors define the positions of the boundaries of the dynamic systems. The junction matrices define the boundary conditions and the couplings among the dynamic systems at and across these boundaries. It emerges, from the derivation of the matrix equation for the response in this format, that the spatial dimensionality of the dynamic systems has a considerable influence on the wave formalism. The interactions via the various paths at the various terminal positions and across the various boundaries lead to summations (and when appropriate integrations) galore; so much so, that the formalism appears too hopelessly interwoven and entangled to be processed by computers of reasonable capacities, today a *useful* measure for the practicality of a formalism. However, dramatic simplification in this cumbersomeness of the formalism results when the dynamic systems are one-dimensional. Further meaningful

simplification occurs if the dynamic systems are basic -- the propagation in the dynamic systems is described by exponential-like propagators. The response of those complexes that may be modeled by a collection of coupled basic one-dimensional dynamic systems is, therefore, illustrated and is now readily formulated [1-5].

Finally, it is proper to mention the comprehensive review by Hodges and Woodhouse on the theories of noise and vibration transmission in complex structures [8]. In this review many aspects of the subject matter covered herein are discussed. Indeed, much of the material and references presented in this review are relevant to this paper. The discussion of the applicability of the formalism of the acoustical response of complex structures to various other areas of physics, is of particular interest. However, the intended purposes and methodologies of the review do not overlap those intended and developed in this paper.

## MULTI-PATHS

One may recognize that the response at $\{\hat{x}, t\}$ may be contributed to by a number of possible paths and that each path is not only describable by an independent impulse response operator, but that only a portion of the drive may participate in contributing a response via this path. The complex is then modeled by a number of paths as is sketched in Figure 1. With the aid of such modeling one may state that for each path the equation for the response mimics equation (1) so that

$$h^{\nu}(\hat{x} \mid \hat{x}', t \mid t')\, p_{e\nu}(\hat{x}', t') = p^{\nu}(\hat{x}, t) \quad , \tag{5}$$

where $\nu$ designates a unique path. [cf. equation (4).] It is helpful and instructive, as the theoretical development proceeds, to exemplify and contrast the derived equations with those pertaining to a simple model. Therefore, the derived equations will be followed immediately by the corresponding equations that are appropriate for the simple model of the complex. The simple model may also be employed to render the equations more explicit. Brief references to this procedure will be made at the appropriate locations in the text. Thus, to begin this procedure, if the

traversal of a path may be specified merely by a time delay $\tau^{\nu}$, the impuse response operator $h^{\nu}(\hat{x} \mid \hat{x}', t \mid t')$ in equation (5) may be expressed more explicitly in the form

$$h^{\nu}(\hat{x} \mid \hat{x}', t \mid t') = \bar{h}^{\nu}(\hat{x} \mid \hat{x}', t \mid t') \, \Delta(\tau^{\nu}) \quad , \tag{6a}$$

where $\bar{h}^{\nu}(\hat{x} \mid \hat{x}', t \mid t')$ plays a secondary role and is functionally reasonable and available, $\Delta(\tau)$ is a temporal operator defined by

$$\Delta(\tau) f(t) = f(t-\tau) \; ; \quad \prod_{n}^{N} \Delta(\tau_n) = \Delta\Big(\sum_{n}^{N} \tau_n\Big) \quad , \tag{7a}$$

the path $\nu$ is assumed unique, the subscript and superscript ($\nu$) identifies the path, $\tau^{\nu}$ designates the elapsed time (retarded time); the time taken by a response induced at $\hat{x}'$ to transfer, via path $\nu$, and arrive at $\hat{x}$, and finally, f is an arbitrary, but well behaved, function of time. The elapsed time $\tau^{\nu}$ is a function of $\hat{x}$ and $\hat{x}'$ and is independent of the temporal variables; $\tau^{\nu} = \tau^{\nu}(\hat{x} \mid \hat{x}')$ but not $\tau^{\nu}(\hat{x} \mid \hat{x}', t \mid t')$. A basic example of the impulse response operator in which $\bar{h}^{\nu}(\hat{x} \mid \hat{x}', t \mid t')$ is more explicitly stated is

$$\begin{aligned} h^{\nu}(\hat{x} \mid \hat{x}', t \mid t') &= \bar{h}^{\nu}(\hat{x} \mid \hat{x}', t \mid t') \, \Delta(\tau^{\nu}) \; ; \\ \bar{h}^{\nu}(\hat{x} \mid \hat{x}', t \mid t') &= \bar{\bar{h}}^{\nu}(\underset{\sim}{x} \mid \underset{\sim}{x}') \, \delta^{0t}(t-t') \; ; \\ \bar{\bar{h}}^{\nu}(\hat{x} \mid \hat{x}') &= \int \bar{g}^{\nu}(\hat{x} \mid \hat{x}') \, d\hat{x}' \cdots \; ; \quad \delta^{0t} = \int \delta(t-t') \, dt' \cdots \end{aligned} \tag{6b}$$

where $\bar{g}^{\nu}(\hat{x} \mid \hat{x}')$ is independent of the temporal variables, and

$$\delta^{0t}(t-t') \, f(t') = f(t) \; ; \quad \prod_{n=1}^{N} \delta^{0t}(t_{n+1} - t_n) = \delta^{0t}(t_{N+1} - t_1) \quad . \tag{7b}$$

[cf. equation (4).] To cast equation (5) in the format of equation (1) one needs to construct the impulse response operator and the drive in vectorial forms; i.e.,

$$\hat{h}(\hat{x} \mid \hat{x}', t \mid t') = \sum_{\nu} h^{\nu}(\hat{x} \mid \hat{x}', t \mid t') \, \hat{\nu} \quad , \tag{8a}$$

$$\hat{p}_e(\hat{x}', t') = \sum_\sigma \hat{\sigma}\, p_{e\sigma}(\hat{x}', t') \ ; \qquad [\hat{\nu}\hat{\sigma}] = [\hat{\sigma}\hat{\nu}] = \delta_{\nu\sigma} \ , \tag{9}$$

where again the caret atop a quantity signifies its vectorial nature.[1,2] [cf. equations (1a) and (1g).] For the simple model, equation (8a) assumes the form

$$\hat{h}(\hat{x} \mid \hat{x}', t \mid t') = \sum_\nu \overline{h}^\nu(\hat{x} \mid \hat{x}', t \mid t')\, \Delta(\tau^\nu)\, \hat{\nu} \ . \tag{8b}$$

The grand equation for the response is then derived from equations (1), (4), (8), and (9) in the form

$$[\hat{h}(\hat{x} \mid \hat{x}', t \mid t')\, \hat{p}_e(\hat{x}', t')] = p(\hat{x}, t) \ ; \qquad p(\hat{x}, t) = \sum_\nu p^\nu(\hat{x}, t) \ . \tag{10}$$

It is noted that a path is defined either in terms of components in the impulse response operator, in the drive, or in both. Although not exactly in this format, equations (5) and (10) have proved useful in handling problems relating to structural response. Problems relating to propagations at sea have made greater use of such decompositions than structural vibrations have. The reason for the limited deployment of this formalism in solving problems related to structural vibrations, stems from the difficulty and cumbersomeness that often still remains in the description of the partial impulse response operator $h^\nu(\hat{x} \mid \hat{x}', t \mid t')$ – –the impulse response operator describing the transfer via a single unique path; path $\nu$. This statement holds true even for the partial impulse response operator of models that are simple. [The simplicity of these models may arise because they represent simple practical structural complexes and not necessarily because they represent excessively simplified complexes.] Much of the progress made has been linked with statistical measures designed to perform the summation over the paths; e.g., by ensemble averaging. In

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[2]*Paths (and, when approapriate, positions at junctions) are cast in terms of sums over discrete values. However, in principle these sums may be continuous so that the summations may be appropriately replaced, in part or in whole, by integrations. This transition from the discrete to the continuous may be interpreted from ray, via beam, to wave descriptions. In this sense, in the text, the reference to wave propagation is more akin with ray propagation.*

these cases the partial impulse response operators need not be precisely defined and described. Averaging of this kind has been applied with some success to limited and simplified situations in structural vibrations; e.g. References 9 and 10. On the other hand, averaging of this kind has been traditionally quite acceptable in solving problems relating to propagations at sea; e.g. References 11 and 12. On a typical basis, statistical measures may yield from equation (10) a simplified equation (1). However, in this paper, rather than investigate the restrictions that are imposed on, and explore the advantages that are accrued from the resulting equation, one may *inquire* whether alternate and/or further analytical decompositions exist from which more profound benefits may be attained.

## MULTI-DYNAMIC SYSTEMS

One may attempt to subdivide the complex into a number of coupled dynamic systems as is sketched in Figure 2. [cf. Figure 1.] A contrived model may then be constructed in which the dynamic systems are rendered uncoupled. The uncoupling of the dynamic systems may be instituted, for example, by artificially isolating or extrapolating the dynamic systems. In this contrived model each dynamic system admits to an impulse response function that is hopefully reasonable to describe. The equation for the response of the contrived model of the complex, in which the dynamic systems are rendered, artificially if necessary, uncoupled, may be cast in the form

$$\underset{\sim}{h}_u (\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', t \,|\, t') \; \underset{\sim}{p}_e (\hat{\underset{\sim}{x}}', t') = \underset{\sim}{p}_u (\hat{\underset{\sim}{x}}, t) \quad , \tag{11a}$$

where

$$\hat{\underset{\sim}{x}} = \{\hat{x}_j\} \; ; \quad \hat{\underset{\sim}{x}}' = \{\hat{x}'_i\} \; ; \quad \hat{x}_j = \sum_s^{S_0} \hat{s}\, x_{sj} \quad ; \quad d\hat{x}'_i = \prod_s^{S_0} d\hat{x}'_{si} \quad , \tag{12}$$

$$\underset{\approx}{h}_u(\hat{\underset{\sim}{x}} | \hat{\underset{\sim}{x}}', t | t') = \left( h_{uj}(\hat{x}_j | \hat{x}_j', t | t') \, \delta_{ji} \right) \quad , \tag{13a}$$

$$\underset{\sim}{p}_e(\hat{\underset{\sim}{x}}', \hat{t}') = \{ p_{ei}(\hat{x}_i', t') \} \; ; \qquad \underset{\sim}{p}_u(\hat{\underset{\sim}{x}}, t) = \{ p_{uj}(\hat{x}_j, t) \} \quad . \tag{14a}$$

$p_{ei}\,(\hat{x}_i', \; t')$ is the drive applied at positions $\{\hat{x}_i', \; t'\}$ to the (i)th dynamic system, $h_{uj}\,(\hat{x}_j | \hat{x}_j', t | t')$ is the impulse response operator of the uncoupled (j)th dynamic system, and $p_{uj}\,(\hat{x}_j, t)$ is the response at position $\{\hat{x}_j, \; t\}$ in the (j)th dynamic system. In component form equation (11a) is

$$h_{uj}(\hat{x}_j | \hat{x}_j', t | t') \, p_{ej}(\hat{x}_j', t') = p_{uj}(\hat{x}_j, t) \quad . \tag{15a}$$

It is noted that the dynamic systems need not span the same spatial dimensionality. Thus, $\hat{x}_j$ may be of different spatial dimensionality than that of $\hat{x}_i$ with $i \neq j$.

The insitu model of the complex must account for the various couplings that may exist between the multiple dynamic systems. Indeed, the specification of the couplings is crucial in this modeling. It is proposed that, in turn, the modeling may reveal the manner in which the couplings need be specified. This interaction between the modeling and the specification of the couplings is essential to the success of this formalism. [It is to be understood that in the specification of the couplings are included deviations in the boundary conditions under which equation (11a) is stated. Therefore, the specification of the couplings may cause changes in the definitions of some dynamic systems without, necessarily, causing changes in the fact that they are uncoupled; i.e., some dynamic systems may be uncoupled in the insitu model.] Taking account of the couplings, one may formally state the equation for the response in the matrix form

$$\underset{\approx}{h}(\hat{\underset{\sim}{x}} | \hat{\underset{\sim}{x}}', t | t') \, \underset{\sim}{p}_e(\hat{\underset{\sim}{x}}', t') = \underset{\sim}{p}(\hat{\underset{\sim}{x}}, t) \quad , \tag{11b}$$

or equivalently in the component form

$$\sum_i h_{ji}(\hat{x}_j \mid \hat{x}'_i, t \mid t')\, p_{ei}(\hat{x}'_i, t') = p_j(\hat{x}_j, t) \quad , \tag{15b}$$

where

$$\underset{\approx}{h}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') = \left( h_{ji}(\hat{x}_j \mid \hat{x}'_i, t \mid t') \right) \quad , \tag{13b}$$

$$\underset{\sim}{p}_e(\hat{\underset{\sim}{x}}', t') = \{p_{ei}(\hat{x}'_i, t')\} \quad ; \quad \underset{\sim}{p}(\hat{\underset{\sim}{x}}, t) = \{p_j(\hat{x}_j, t)\} \quad . \tag{14b}$$

[cf. equations (11a), (15a), (13a), and (14a), respectively.] In these equations, the impulse response operator $h_{ji}(\hat{x}_j \mid \hat{x}'_i, t \mid t')$ is the "transfer operator" from the drive position at $\{\hat{x}'_i, t'\}$ in the (i)th dynamic system to the observation position at $\{\hat{x}_j, t\}$ in the (j)th dynamic system, $p_{ei}(\hat{x}'_i, t')$ is the external drive at $\{\hat{x}'_i, t'\}$ on the (i)th dynamic system, and $p_j(\hat{x}_j, t)$ is the response at $\{\hat{x}_j, t\}$ in the (j)th dynamic system due to all the external drives in the various dynamic systems. One may also cast equation (11b) in the more decomposed form

$$\underset{\approx}{h}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t')\, \underset{\approx}{p}_e(\hat{\underset{\sim}{x}}', t') = \underset{\approx}{p}(\hat{\underset{\sim}{x}}, t) \quad , \tag{11c}$$

where the (external) drive matrix and the response matrix are defined

$$\underset{\approx}{p}_e(\hat{\underset{\sim}{x}}', t') = \left( p_{ei}(\hat{x}'_i, t')\, \delta_{ik} \right) \quad ; \quad \underset{\approx}{p}(\hat{\underset{\sim}{x}}, t) = \left( p_{ji}(\hat{x}_j, t) \right) \quad , \tag{14c}$$

respectively. In component form equation (11c) reads

$$h_{ji}(\hat{x}_j \mid \hat{x}'_i, t \mid t')\, p_{ei}(\hat{x}'_i, t') = p_{ji}(\hat{x}_j, t) \quad . \tag{15c}$$

[cf. equations (15a) and (15b).] It is obvious that when the dynamic systems are artificially rendered more and more uncoupled, in the prescribed manner, the result is

$$\underset{\approx}{h}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') \Rightarrow \underset{\approx}{h}_u(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') \quad . \tag{16}$$

The inverse of equation (16) is more difficult and cumbersome to establish. Not only off diagonal elements appear, rather than disappear, but the diagonal elements undergo changes. Therefore, the establishment of the uncoupled impulse response matrix operator $\underline{\underline{h}}_u(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t')$ may not imply a direct and an immediate step in the process of deriving the impulse response matrix operator $\underline{\underline{h}}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t')$. Nonetheless, situations may arise in which such an establishment may constitute a first step in the process of deriving $\underline{\underline{h}}$; e.g., $\underline{\underline{h}}_u$ may be considered the zeroth order approximation to $\underline{\underline{h}}$. In this context modeling the complex in terms of multiple dynamic systems necessitates specifying the couplings among the dynamic systems in a manner that is compatible with the definition of the contrived model of uncoupled dynamic systems.

The derivation of the impulse response matrix operator $\underline{\underline{h}}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t')$ for the subdivided complex can be achieved by a number of different approaches. Invariably in these approaches further analytical decompositions are instituted. Are these further analytical decompositions of help in reducing the difficulty and cumbersomeness of ascertaining the response of structural complexes excited by complex external driving systems? Two distinct approaches are subsequently discussed, and comments are made with respect to the answer to this question.

## MODAL APPROACH

An analytical decomposition that is widely familiar is the modal analysis. A model in which the multiple dynamic systems are rendered uncoupled by isolation is amenable, by its very structure, to modal analysis. Indeed, the isolated dynamic systems in the complex are conveniently chosen in a manner that make them, individually and collectively, suitable to a modal analysis. The model is sketched in Figure 3. [cf. Figure 1 and 2.] The equation for the response is then typified by equation (11a). It is assumed that each appropriately isolated dynamic system is characterized by an impedance operator. The impedance operator $z_j(\hat{x}_j, t)$ is the inverse of the impulse response operator $h_{uj}(\hat{x}_j \mid \hat{x}_j', t \mid t')$ of the (j)th dynamic system. [cf. equation (13a).] It

is further assumed that the chosen impedance operators are eigen-impedance operators. It is noted that the definition of the eigen-impedance operators is not unique; there are usually a number of ways in which the dynamic systems in the complex may be isolated. A convenient set of eigen-impendance operators is then selected to define the model of the complex. Each eigen-impedance operator defines a closed (complete) set of orthogonal (normal) eigenfunctions and a corresponding set of eigen-values. The modal (eigen-) equations for the response, the closure, and orthogoraliity are expressed by

$$z_j(\hat{x}_j, t)\, \phi_{jn}(\hat{x}_j) = z_{jn}(t)\, \phi_{jn}(\hat{x}_j) \quad , \tag{17a}$$

$$\sum_n \phi_{jn}(\hat{x}_j)\, \phi_{jn}(\hat{x}'_j) = \delta(\hat{x}_j - \hat{x}'_j) \quad ; \tag{17b}$$

$$\int \phi_{jn}(\hat{x}_j)\, d\hat{x}_j\, \phi_{jm}(\hat{x}_j) = \delta_{nm} \quad , \tag{17c}$$

respectively, where $z_j(\hat{x}_j, t)$ designates the eigen-impedance operator, $\phi_{jn}(\hat{x}_j)$ designates an eigenfunction, and $z_{jn}(t)$ designates an eigen-value for the (j)th dynamic system, n (and m) is the modal (eigen–) index, and

$$\delta(\hat{x}_j - \hat{x}'_j) = \prod_s^{s_0} \delta(x_{sj} - x'_{sj}) \quad . \tag{18}$$

Strictly the modal index need be cast in terms of a vector. That is, the index n (and m) in the equations in this section need be replaced by the vector index $\hat{n}$ (and $\hat{m}$), where $\hat{n} = \sum_{s=1}^{s_0} \hat{s}\, n_s$. Since the modal approach is widely familiar, such, and other, details in the modal approach are not dwelled upon in this paper. From equations (11), (14), and (17) one may state and readily obtain

$$p_{ei}(\hat{x}'_i, t') = \sum_m p_{eim}(t')\, \phi_{im}(\hat{x}'_i) \quad ;$$
$$p_j(\hat{x}_j, t) = \sum_n p_{jn}(t)\, \phi_{jn}(\hat{x}_j) \quad , \tag{19a}$$

where

$$p_{eim}(t') = \int p_{ei}(\hat{x}'_i, t')\, d\hat{x}'_i\ \phi_{im}(\hat{x}'_i)\ ;$$
$$p_{jn}(t) = \int p_{jn}(\hat{x}_j, t)\, d\hat{x}_j\ \phi_{jn}(\hat{x}_j)\ , \tag{19b}$$

The insitu model is now constructed by introducing the couplings among the dynamic systems. The couplings are compatibly defined in terms of coupling impedances. An impedance matrix equation for the response may then be defined in the form

$$\underset{\approx}{z}\, \underset{\sim}{p}(\underset{\sim}{\hat{x}}, t) = \underset{\sim}{p_e}(\underset{\sim}{\hat{x}}, t)\ ; \quad \underset{\sim}{p}(\underset{\sim}{\hat{x}}, t) = \{p_j(\hat{x}_j, t)\}\ ; \quad \underset{\sim}{p_e}(\underset{\sim}{\hat{x}}, t) = p_{ej}(\hat{x}_j, t)\ ;$$
$$\underset{\approx}{z} = \left(z_j(\hat{x}_j, t)\, \delta_{ji} - \int z_{ji}(\hat{x}_j | \hat{x}'_i, t)\, d\hat{x}'_i \int \delta(\hat{x}'_i - \hat{x}_i)\, d\hat{x}_i \cdots\right)\ , \tag{20}$$

where $z_j(\hat{x}_j, t)$ is as defined in equation (17) and $z_{ji}(\hat{x}_j | \hat{x}'_i, t)$ is the coupling impedance, this impedance describes the coupling between the (i)th and the (j)th dynamic systems. [It is noted that to render the dynamic systems isolated; equation (16), one merely removes the coupling impedances; or more generally, one renders $\underset{\approx}{z}$ diagonal.] Transforming equation (20) into the modal domain and making use of the orthogonality of the eigenfunctions yields

$$\underline{\underline{z}}(t)\ \underline{p}(t) = \underline{p}_e(t)\ , \tag{21a}$$

$$\underline{p}_e(t') = \{\cdots \underline{p}_{ei}(t') \cdots\}\ ; \quad \underline{p}_{ei}(t') = \{p_{eim}(t')\}\ ;$$
$$\underline{p}(t) = \{\cdots \underline{p}_j(t) \cdots\}\ ; \quad \underline{p}_j(t) = \{p_{jn}(t)\}\ , \tag{22a}$$

$$\underline{\underline{z}} = \left(z_{jn}(t)\, \delta_{ji}\, \delta_{nm} - z_{jnim}(t)\right)\ , \tag{22b}$$

where $z_{jn}(t)$ is defined in equation (17a) and

$$z_{jnim}(t) = \int \phi_{jn}(\hat{x}_j)\, d\hat{x}_j\ z_{ji}(\hat{x}_j | \hat{x}'_i, t)\, d\hat{x}'_i\ \phi_{im}(\hat{x}'_i)\ . \tag{23a}$$

or equivalently

$$z_{ji}(\hat{x}_j | \hat{x}'_i, t) = \sum_n \sum_m \phi_{jn}(\hat{x}_j)\; z_{jnim}(t)\; \phi_{im}(\hat{x}'_i) \quad . \tag{23b}$$

The quantity $z_{jnim}(t)$ is the coupling eigen-value describing the modal coupling between the (m)th mode in the (i)th dynamic system and the (n)th mode in the (j)th dynamic system. In this description, $z_{jnjn}$ is the self-coupling eigen-value; it is these eigen-values that modify the diagonal elements when couplings are introduced. Equation (21) may be inverted so that the equation for the response is expressed in terms of the impulse response matrix operator $\underline{\underline{h}}$ that is associated with $\underline{\underline{z}}$. The inverted equation, in matrix form, is

$$\begin{aligned} &\underline{p}(t) = \underline{\underline{h}}(t|t')\;\underline{p}_e(t') \;; \qquad \underline{\underline{h}}(t|t') = \int \underline{\underline{g}}(t|t')\,dt' \cdots \;; \\ &\underline{\underline{h}}(t|t') = \big(h_{jnim}(t|t')\big) \;; \qquad \underline{\underline{g}}(t|t') = \big(g_{jnim}(t|t')\big) \;, \end{aligned} \tag{21b}$$

or equivalently, in component form, is

$$\begin{aligned} &\sum_i \sum_m h_{jnim}(t|t')\, p_{eim}(t') = p_{jn}(t) \;; \\ &h_{jnim}(t|t') = \int g_{jnim}(t|t')\,dt' \cdots \;, \end{aligned} \tag{21c}$$

where

$$g_{jnim}(t|t') = \int \phi_{jn}(\hat{x}_j)\, d\hat{x}_j\;\, g_{ji}(\hat{x}_j | \hat{x}'_i, t | t')\, d\hat{x}'_i\; \phi_{im}(\hat{x}'_i) \quad . \tag{24a}$$

or equivalently

$$g_{ji}(\hat{x}_j | \hat{x}'_i, t | t') = \sum_n \sum_m \phi_{jn}(\hat{x}_j)\; g_{jnim}(t|t')\, \phi_{im}(\hat{x}'_i) \quad . \tag{24b}$$

[cf. equations (15b) and (23).] The inversion of equation (21a) into equation (21b), and to that matter, vice versa, is greatly facilitated if the dynamic systems and the couplings among them remain unchanged with time — the dynamic systems and the couplings among them are temporarily stationary. That is, $\underline{\underline{z}}(t)$ is pure differential operator in the temporal domain and

$\underline{\underline{g}}(t\,|\,t') = (2\pi)^{-1/2}\,\underline{\underline{g}}(t-t')$. [cf. equation (3a).] Assuming this temporal stationarity and performing Fourier transformations with respect to the temporal variable on equations (21a), (21b), (23), and (24) yield

$$\tilde{\underline{\underline{z}}}(\omega)\;\tilde{\underline{p}}(\omega) = \tilde{\underline{p}}_e(\omega)\;; \qquad \tilde{\underline{\underline{z}}}(\omega) = \left(\tilde{z}_{jnim}(\omega)\right) \quad , \tag{25a}$$

$$\tilde{\underline{p}}(\omega) = \tilde{\underline{\underline{g}}}(\omega)\,\tilde{\underline{p}}_e(\omega)\;; \qquad \tilde{\underline{\underline{g}}}(\omega) = \left[\tilde{\underline{\underline{z}}}(\omega)\right]^{-1} \quad , \tag{25b}$$

$$\tilde{z}_{ji}(\hat{x}_j\,|\,\hat{x}'_i,\omega) = \sum_{\underline{n}}\sum_{\underline{m}}\;\phi_{jn}(\hat{x}_j)\;\tilde{z}_{jnim}(\omega)\;\phi_{im}(\hat{x}'_i) \quad , \tag{23c}$$

$$\tilde{g}_{ji}(\hat{x}_j\,|\,\hat{x}'_i,\omega) = \sum_{\underline{n}}\sum_{\underline{m}}\;\phi_{jn}(\hat{x}_j)\;\tilde{g}_{jnim}(\omega)\;\phi_{im}(\hat{x}'_i) \quad , \tag{24c}$$

respectively, where $\tilde{\underline{p}}(\omega)$, $\tilde{\underline{p}}_e(\omega)$, and $\tilde{\underline{\underline{g}}}(\omega)$ are the Fourier transforms of $\underline{p}(t)$, $\underline{p}_e(t)$, and $(2\pi)^{-1/2}\,\underline{\underline{g}}(t-t')$, respectively, and $\tilde{\underline{\underline{z}}}(\omega)$ is the Fourier eigen-value matrix of the temporal matrix operator $\underline{\underline{z}}(t)$; namely,

$$\underline{\underline{z}}(t)\left((2\pi)^{-1/2}\exp(i\omega t)\,\delta_{ji}\,\delta_{nm}\right) = \tilde{\underline{\underline{z}}}(\omega)\left((2\pi)^{-1/2}\exp(i\omega t)\,\delta_{ji}\,\delta_{nm}\right) \quad . \tag{26}$$

It is noted that the inversion from equation (25a) into equation (25b), and vice versa, is algebraic. The simplicity of the inversion, in the frequency domain, when a temporal stationarity prevails, is thus clearly demonstrated. However, as equation (25a) indicates, the formalism just discussed is predicated not only on the intimate knowledge of the impedance operators that describe the isolated dynamic systems, but also on knowledge of the coupling impedance operators. Such knowledge is rarely available. In this connection it may be noted that the eigenfunctions need not be directly defined by an actual impedance operator. For example, eigenfunctions that are geometrically

defined are quite acceptable in deriving the equations in this section. However, the geometry, like the spatial dependence of the eigen-impedance operators, needs to remain stationary with respect to the temporal variable. Using artificial eigenfunctions is not without penalty, it renders the modal quantities and parameters also artificial and, therefore, not physically interpretable.

Comparing equation (15) with equation (21) it is observed that the spatial dependence is stripped off by the modal procedure; a simplification indeed, however, at some expense. The rank of the matrix $\underline{\underline{h}}$ is swelled from that of $\underset{\approx}{h}$, defined in equation (13), by the number of modes within the complex. Even in simple complexes, such swelling may be quite substantial and equation (21) and (25) may, thus, become unwieldy. In addition, seldom is knowledge of the natural eigenfunctions readily available, even in these simple complexes. Means to reduce the rank of equation (21) and (25) may prove essential and the requirement for knowledge of the eigenfunctions (and the eigen-values of the coupling impedance operators) may need relaxation. Again, statistical means may have to be devised to induce such a reduction and to suppress the need to know detail forms of the eigenfunctions of the isolated dynamic systems and the eigen-values that specify the couplings among the dynamic systems of the complex. The statistical energy analysis (SEA) is one such a device [13, 14]. The rendering of SEA from equation (21) and/or (25) lies, however, beyond the scope of this paper.

## WAVE APPROACH

Another approach to an analytical decomposition is a wave propagation approach. This approach is not new; it has been employed in the vibrational analysis of complex structures [8]. Indeed, in part, this approach is employed in Section II. In this paper, however, an attempt is made to systematize this approach and finally recover and extend some recent work by the authors [4,5]. Equations (5) and (11) are considered basic. In this approach the dynamic systems in the contrived model are rendered uncoupled by extrapolation. An uncoupled dynamic system in the

complex is chosen so that a propagation mode can be assigned to it; namely, were the (j)th dynamic system extrapolated boundlessly, the propagating path $\nu$ in it can be defined in terms of the propagator function $h^{\nu}_{\infty j}(\hat{x}_j \mid \hat{x}'_j, t \mid t')$ so that a corresponding external drive $p_{ev j}(\hat{x}'_j, t')$ would induce a response $p^{\nu}_{\infty j}(\hat{x}_j, t)$ at the position $\{\hat{x}_j, t\}$ on the extrapolated dynamic system. The analytical expression for this process is

$$h^{\nu}_{\infty j}(\hat{x}_j \mid \hat{x}'_j\,,\ t \mid t')\, p_{ev j}(\hat{x}'_j\,, t') = p^{\nu}_{\infty j}(\hat{x}_j\,, t) \quad . \tag{27}$$

[cf. equations (5) and (15a).] Again, were the path of propagation a simple path, in equation (27) $h^{\nu}_{\infty j}(\hat{x}_j \mid \hat{x}'_j\,,\ t \mid t')$ assumes the more explicit form

$$h^{\nu}_{\infty j}(\hat{x}_j \mid \hat{x}'_j\,,\ t \mid t') = \bar{h}^{\nu}_{\infty j}(\hat{x}_j \mid \hat{x}'_j\,,\ t \mid t')\, \Delta^{\nu}_j \quad , \tag{28}$$

$$\Delta^{\nu}_j \equiv \Delta(\tau^{\nu}_{jj}) \quad , \tag{29}$$

where the quantity $\tau^{\nu}_{jj}$ designates the time taken by a response induced at $\hat{x}'_j$ to transfer, via path $\nu$ on the extrapolated dynamic system, and arrive at $\hat{x}_j$ ; $\tau^{\nu}_{jj} = \tau^{\nu}_{jj}(\hat{x}_j \mid \hat{x}'_j)$ . [cf. equation (5) through (7).] The extrapolation is considered proper provided it is instituted in a manner that would avoid back-scattering, from regions outside, into regions occupied by the actual dynamic system. It is noted that the extrapolation is not unique; there are several ways in which it can be instituted. However, an extrapolation of this kind is essential and central to the wave approach here proposed. Equation (27) for the multitude of dynamic systems may be stated in the matrix form

$$\underset{\sim}{h}^{\nu}_{\infty}(\underset{\sim}{\hat{x}} \mid \underset{\sim}{\hat{x}}', t \mid t')\, \underset{\sim}{p}_{ev}(\underset{\sim}{\hat{x}}', t') = \underset{\sim}{p}^{\nu}_{\infty}(\underset{\sim}{\hat{x}}, t) \quad , \tag{30a}$$

where

$$\underset{\approx}{h}^{v}_{\infty}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t') = \left(h^{v}_{\infty j}(\hat{x}_j|\hat{x}'_j, t|t')\,\delta_{ji}\right) \quad , \tag{31a}$$

$$\underset{\sim}{p}_{ev}(\hat{\underset{\sim}{x}}', t') = \{p_{evj}(\hat{x}'_i, t')\} \ ; \quad \underset{\sim}{p}^{v}_{\infty}(\hat{\underset{\sim}{x}}, t) = \{p^{v}_{\infty j}(\hat{x}_j, t)\} \quad . \tag{32}$$

In the wave approach, equation (30) is equivalent to that of the generic expression specified in equation (11a). [The prescribed "extrapolation" in the wave approach plays the role of the "isolation" in the modal approach discussed in the preceding section. These two prescriptions are designed to derive equation (11a) in forms that are fundamental to each of the two approaches.] From equations (27), (28), and (32) one obtains

$$\underset{\approx}{\bar{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t')\ \underset{\approx}{\Delta}(\underset{\sim}{\tau}^{v})\ \underset{\sim}{p}_{ev}(\hat{\underset{\sim}{x}}', t') = \underset{\sim}{p}^{v}_{\infty}(\hat{\underset{\sim}{x}}, t) \quad . \tag{30b}$$

where

$$\underset{\approx}{\bar{h}}^{v}_{\infty}(\hat{x}_j|\hat{x}'_j, t|t') = \left(\bar{h}^{v}_{\infty j}(\hat{x}_j|\hat{x}'_j, t|t')\,\delta_{ji}\right) \quad , \tag{31b}$$

$$\prod_{n}^{N} \underset{\approx}{\Delta}(\underset{\sim}{\tau}_n) = \underset{\approx}{\Delta}\left(\sum_{n}^{N}\underset{\sim}{\tau}_n\right) \ ; \ \underset{\approx}{\Delta}(\underset{\sim}{\tau}^{v}) = \left(\Delta^{v}_j \ \ \delta_{ji}\right) \ ; \ \Delta^{v}_j = \Delta(\tau^{v}_{jj}) \ ; \ \underset{\sim}{\tau}^{v} = \{\tau^{v}_{jj}\} \quad . \tag{33}$$

[cf. equations (5), (7), (11), and (29).] The matrix operator $\underset{\approx}{\Delta}$ can accommodate a vector dependent variable because it is, by definition, a diagonal matrix. The vectorial form of the elapsed time $\underset{\sim}{\tau}^{v}$ is simpler and, therefore, more convenient than its diagonal matrix $\underset{\approx}{\tau}^{v}$ counterpart would be. [cf. equations (11b) and (11c).] In the wave approach the couplings are compatibly defined in terms of a set of junctions. The junctions define the insitu boundaries and the interconnections among the dynamic systems. A terminal position at a boundary (a junction) in the (j)th dynamic

designated $\hat{x}_{aj}$. A terminal position vector may then be defined

$$\underset{\sim}{\hat{x}}_a = \{\hat{x}_{aj}\} \; . \tag{34}$$

The set of the terminal position vectors define the extents and the boundaries of the dynamic systems in the complex. To complete the definition of the model of the complex it is necessary to define the junction matrices

$$\underset{\approx}{T}^{v}_{b\sigma} = \left(T^{vk}_{b\sigma ji}\right) \; . \tag{35a}$$

The junction matrix element $T^{vk}_{b\sigma ji}$ defines the conversion, partially or totally, of an incident wave propagating in path $\sigma$, on a junction at the terminal position $\hat{x}_{bi}$ in the (i)th dynamic system, into a wave that emerges and propagates, in path $v$, away from the junction at the initial position $\hat{x}_{ki}$ in the (j)th dynamic system, see Figure 4b. The conversion is accompanied also by a temporal integral operation. Stating the process just discussed more explicitly, with minimal loss in generality, one may cast[3]

$$\underset{\approx}{T}^{vk}_{b\sigma} \equiv \underset{\approx}{y}_k(\underset{\sim}{\hat{x}}''' \,|\, \underset{\sim}{\hat{x}}_k)\, \underset{\approx}{\Lambda}^{vk}_{b\sigma}\, \underset{\approx}{y}^{0b}(\underset{\sim}{\hat{x}}_b \,|\, \underset{\sim}{\hat{x}}') \; , \tag{35b}$$

where it is assumed that the spatial and temporal operative properties of the junction matrix can be separated; $\underset{\approx}{\Lambda}^{vk}_{b\sigma}$ is a temporal integral operator and $y^{0b}$ is a spatial integral operator

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[3]*With certain modifications and extensions in the formalism, one may account for situations in which the external drives may be placed also within the junctions. Such modifications and extensions were introduced in Reference 2. However, such considerations lie beyond the scope of the present paper.*

$$\underset{\approx}{\Lambda}^{vk}_{b\sigma}(t''\,|\,t''')\, f_e(t''') = \underset{\approx}{\Lambda}^{vk}_{b\sigma}(0)\, f(t'') \;;$$

$$\underset{\approx}{y}^{0b}(\underset{\sim}{\hat{x}}_b\,|\,\underset{\sim}{\hat{x}}'') = \int \underset{\approx}{y}^{b}(\underset{\sim}{\hat{x}}_b\,|\,\underset{\sim}{\hat{x}}'')\, d\underset{\approx}{\hat{x}}'' \cdots \;; \quad d\underset{\approx}{\hat{x}}'' = \left(d\hat{x}_j''\,\delta_{ji}\right) \;, \tag{36a}$$

where $\underset{\approx}{\Lambda}^{vk}_{b\sigma}(0)$ is merely an algebraic matrix factor, $f_e$ and $f$ are arbitrary, but well behaved, functions of the (dummy) temporal variables $t''$ and $t'''$, respectively, and $\underset{\approx}{y}^{b}$ and $\underset{\approx}{y}_k$ are algebraic matrices in the (dummy) spatial vector variables $\underset{\sim}{\hat{x}}''$ and $\underset{\sim}{\hat{x}}'''$ and the terminal position vectors $\underset{\sim}{\hat{x}}_b$ and $\underset{\sim}{\hat{x}}_k$. To exhibit the properties of the junction matrix even more explicitly, the simple model is used yet again. For this model the junction matrix is defined in the form

$$\underset{\approx}{T}^{vk}_{b\sigma}(\underset{\approx}{\tau}^{vk}_{b\sigma}) = \underset{\approx}{\delta}_k\, \underset{\approx}{\Lambda}^{vk}_{b\sigma}\, \underset{\approx}{\delta}^{0b} \;, \tag{35c}$$

where

$$\underset{\approx}{\Lambda}^{vk}_{b\sigma} = \left(\Lambda^{vk}_{b\sigma ji}(\tau^{vk}_{b\sigma ji})\, \delta^{0t}\right) \;, \tag{36b}$$

$$\Lambda^{vk}_{b\sigma ji}(\tau^{vk}_{b\sigma ji}) = \Lambda^{vk}_{b\sigma ji}(0)\;\; \Delta(\tau^{vk}_{b\sigma ji}) \;,$$

$$\delta^{0t} = \int \delta(t'' - t''')\, dt''' \cdots \;, \tag{36c}$$

$$\underset{\approx}{\delta}_k = \underset{\approx}{\delta}(\underset{\sim}{\hat{x}}''' - \underset{\sim}{\hat{x}}_k) = \left(\delta(\hat{x}_j''' - \hat{x}_{kj})\,\delta_{ji}\right) \;;$$

$$\underset{\approx}{\delta}^{0b} = \int \underset{\approx}{\delta}(\underset{\sim}{\hat{x}}_b - \underset{\sim}{\hat{x}}'')\, d\underset{\approx}{\hat{x}}'' \cdots \;; \quad d\underset{\approx}{\hat{x}}'' = \left(d\hat{x}_j''\,\delta_{ji}\right) \;, \tag{36d}$$

$\Lambda^{vk}_{b\sigma ji}(0)$ and $\tau^{vk}_{b\sigma ji}$ are merely algebraic factors. As equation (36c) makes clear, the elements of $\underset{\approx}{\tau}^{vk}_{b\sigma}$

define a set of simple time delays that are imposed on the appropriate waves as they traverse the junctions.

At this stage, it is instructive to note in passing, a differential feature between the initial models for the wave and the modal formalisms. In the wave approach the intial model is defined in terms of properly extrapolated dynamic systems; these dynamic systems are uncoupled. A proper extrapolation is achieved by manipulating the model so that the junction matrices vanish; e.g., rendering the $\underset{\approx}{\Delta}^{vk}_{b\sigma}$'s, in equations (35b) and (35c), identically zero. On the other hand, in the modal approach the initial model is defined in terms of properly isolated dynamic systems; these dynamic systems are uncoupled. A proper isolation is achieved by manipulating the model so that the nondiagonal elements in the junction matrices vanish; e.g., rendering the $\underset{\approx}{\Delta}^{vk}_{b\sigma}$'s, in equations (35b) and (35c), diagonal matrices.

Reiterating then, the model of the complex is defined in terms of the propagator matrix $\underset{\approx}{h}^{v}_{\infty}$, the terminal position vector $\hat{\underset{\sim}{x}}_{a}$, and, finally, the junction matrix $\underset{\approx}{T}^{vk}_{b\sigma}$. Analogous to equations (5) and (11), and with the help of Figure 4 and equations (30) and (32), one may formally cast the equation for the response vector $\underset{\sim}{p}^{v}(\hat{x}, t)$ that is solely in path v - - the partial response vector - - in the matrix form

$$[\hat{\underset{\approx}{h}}^{v}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') \; \hat{\underset{\sim}{p}}_{e}(\hat{\underset{\sim}{x}}', t)] = \underset{\sim}{p}^{v}(\hat{\underset{\sim}{x}}', t) \quad , \tag{37}$$

where the partial impulse response matrix operator is of the form[1]

$$\hat{\underset{\approx}{h}}^{v}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') = \underset{\approx}{h}^{v}_{\infty}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}', t \mid t') \, \underset{\approx}{v}$$
$$+ \sum_{k} \sum_{b} \sum_{\sigma} \underset{\approx}{h}_{\infty}(\hat{\underset{\sim}{x}} \mid \hat{\underset{\sim}{x}}''', t \mid t''') \, \underset{\approx}{T}^{vk}_{b\sigma} \, \hat{\underset{\approx}{h}}^{\sigma}(\hat{\underset{\sim}{x}}'' \mid \hat{\underset{\sim}{x}}', t'' \mid t') \quad , \tag{38a}$$

and the external drive vector is of the form[1]

$$\hat{\underset{\sim}{p}}_e(\hat{\underset{\sim}{x}}, t') = \sum_{\sigma} \underset{\approx}{\sigma}\, \underset{\sim}{p}_{e\sigma}(\hat{\underset{\sim}{x}}', t) \ ; \qquad \underset{\approx}{\sigma} = (\hat{\sigma}_j\, \delta_{ji}) \ . \tag{39}$$

[cf. equations (8) through (10).] With the aid of equation (35b), the partial impulse response matrix operator may be stated in the more informative but abbreviated form

$$\hat{\underset{\approx}{h}}^{\nu} = \underset{\approx}{h}^{\nu}_{\infty}\, \underset{\approx}{\nu} + \sum_{kb\sigma} \underset{\approx}{h}^{\nu}_{\infty}\, \underset{\approx}{y}_k\, \underset{\approx}{\Lambda}^{\nu k}_{b\sigma}\, \underset{\approx}{y}^{0b}\, \hat{\underset{\approx}{h}}^{\sigma} \ . \tag{38b}$$

It is of interest to note that equation (38a & b) is amenable to interpretation in a manner that is commonly employed in structural acoustics; namely, the division of the response into direct and reverberant parts. Of course, there are a number of different ways of defining and cutting this division. In establishing equation (38a & b), however, a division is already suggested. The direct part is to be associated with that part of the response that occurs prior to interactions with any of the junctions. Once the response is contaminated by interactions with a junction, it is portioned to the reverberant response. Reverberation in this context pertains to the initial and subsequent interactions of waves with the junctions of the complex. [Commonly, however, reverberation is reserved to situations in which large multiplicity of this kind of interactions are present; in fact, large enough so that the initial interactions become insignificant.] It is clear that the direct part of the response is associated with the first term on the right of equation (38a & b). This term is dubbed the direct term in the impulse response matrix operator. It is equally clear that the other terms are associated with the reverberant part of the response. These terms are dubbed the reverberant terms in the impulse response matrix operator. The division is then unambiguous and, as can be readily recognized and verified, remains so throughout the wave approach to the derivation of the impulse response operators. The corresponding equations, to equations (38a) and (38b), for the simple model are

$$\hat{\underline{\underline{h}}}^{\nu}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', t \,|\, t') = \bar{\underline{\underline{h}}}^{\nu}_{\infty}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', t \,|\, t') \, \underline{\underline{\Delta}}(\underset{\sim}{\tau}^{\nu}) \, \underline{\underline{\nu}}$$

$$+ \sum_{k} \sum_{b} \sum_{\sigma} \bar{\underline{\underline{h}}}^{\nu}_{\infty}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}''', t \,|\, t''') \, \underline{\underline{\Delta}}(\underset{\sim}{\tau}^{\nu}) \, \underline{\underline{T}}^{\nu k}_{b\sigma} \, \hat{\underline{\underline{h}}}^{\sigma}(\hat{\underset{\sim}{x}}'' \,|\, \hat{\underset{\sim}{x}}', t'' \,|\, t') \quad , \tag{38c}$$

$$\hat{\underline{\underline{h}}}^{\nu} = \underline{\underline{h}}^{\nu}_{\infty} \, \underline{\underline{\nu}} + \sum_{kb\sigma} \underline{\underline{h}}^{\nu}_{\infty} \, \underline{\underline{\delta}}_{k} \, \underline{\underline{\Lambda}}^{\nu k}_{b\sigma} \, \underline{\underline{\delta}}^{0b} \, \hat{\underline{\underline{h}}}^{\sigma} \quad ;$$

$$\underline{\underline{h}}^{\nu}_{\infty} = \bar{\underline{\underline{h}}}^{\nu}_{\infty} \, \underline{\underline{\Delta}}^{\nu} \quad ; \quad \underline{\underline{\Lambda}}^{\nu k}_{b\sigma} = \left( \Lambda^{\nu k}_{b\sigma ji} \, (0) \, \Delta \, (\tau_{b\sigma ji}) \, \delta^{0t} \right) \quad , \tag{38d}$$

respectively. [cf. equations (35c) and (36b) through (36d).] The notations are chosen so that the reduction of equation (38a) to that of equation (38b) may be instituted without confusion; e.g., note the cyclic form in $\underline{\underline{T}}^{\nu k}_{b\sigma}$ with $\sigma \to b \to k \to \nu$; see Figure 4b. In this connection, situations may arise in which the junction matrix $\underline{\underline{T}}^{\nu k}_{b\sigma}$ may be a priori of simpler form. For example, one may find that the terminal position $\hat{\underset{\sim}{x}}_b$ of incidence and the terminal position $\hat{\underset{\sim}{x}}_k$ of departure are the same; e.g., as depicted in Figure 4c. Then $\underline{\underline{T}}^{\nu k}_{b\sigma} \to \underline{\underline{T}}^{\nu b}_{b\sigma} \, \underline{\underline{\delta}}_{kb}$ ; $\underline{\underline{T}}^{\nu b}_{b\sigma} \equiv \underline{\underline{T}}^{\nu}_{b\sigma}$. The summation in equation (38), and in subsequent equations, may then be accordingly and a priori condensed. It may also be noted that the summation over the terminal positions; e.g., in equation (38), implies discreteness. If the discreteness is found to be inappropriate, the summation may, in part or in whole, be replaced by the appropriate integration.[2] It may be proposed, at this stage, that the following notations be recorded for subsequent use

$$\begin{aligned}
\underline{\underline{\Delta}}^{\nu}_{/b} &= \underline{\underline{\Delta}}(\underset{\sim}{\tau}^{\nu}_{/b}) = \left(\Delta(\tau^{\nu}_{/bj})\delta_{ji}\right) \; ; \quad & \tau^{\nu}_{/bj} &= \tau^{\nu}_{j}(x_j \,|\, x_{bj}) \; ; \\
\underline{\underline{\Delta}}^{\nu}_{a/} &= \underline{\underline{\Delta}}(\underset{\sim}{\tau}^{\nu}_{a/}) = \left(\Delta(\tau^{\nu}_{a/j})\delta_{ji}\right) \; ; \quad & \tau^{\nu}_{a/j} &= \tau^{\nu}_{j}(x_{aj} \,|\, x_j) \; ; \\
\underline{\underline{\Delta}}^{\nu}_{ab} &= \underline{\underline{\Delta}}(\underset{\sim}{\tau}^{\nu}_{ab}) = \left(\Delta(\tau^{\nu}_{abj})\delta_{ji}\right) \; ; \quad & \tau^{\nu}_{abj} &= \tau^{\nu}_{j}(x_{aj} \,|\, x_{bj}) \; .
\end{aligned} \tag{40}$$

Equation (38) appears impure in the sense that a number of $\hat{\underline{\underline{h}}}^{\sigma}$'s show in the reverberant terms of the expression for $\hat{\underline{\underline{h}}}$. Means need be devised to purify equation (38). For this purpose equation (38b) is manipulated to construct the relationship

$$\underset{\approx}{B}_{\sigma}^{b}\ \underset{\approx}{y}^{0b}\ \hat{\underset{\approx}{h}}^{\sigma} = \underset{\approx}{y}^{0b}\ \underset{\approx}{h}_{\infty}^{\sigma}\ \underset{\approx}{\sigma} + (\sum_{fe\xi} \underset{\approx}{\Lambda}_{e\xi}^{b\sigma f})\ \underset{\approx}{y}^{0e}\ \underset{\approx}{h}_{\infty}^{\xi}\ \underset{\approx}{\xi} \quad , \tag{41a}$$

where

$$\underset{\approx}{B}_{\sigma}^{b} = [\underset{\approx}{I} - (\sum_{fe\xi} \underset{\approx}{\Lambda}_{e\xi}^{b\sigma f})(\sum_{hc\zeta} \underset{\approx}{\Lambda}_{c\zeta}^{e\xi h}) \sum_{b} \underset{\approx}{\delta}_{cb} \sum_{\sigma} \underset{\approx}{\delta}_{\zeta\sigma}] \quad , \tag{42a}$$

$$\underset{\approx}{\Lambda}_{e\xi}^{b\sigma f} = \underset{\approx}{y}^{0b}\ \underset{\approx}{h}_{\infty}^{\sigma}\ \underset{\approx}{y}_{f}\ \underset{\approx}{\Lambda}_{e\xi}^{\sigma f} \ ; \qquad \underset{\approx}{\delta}_{\zeta\sigma} = (\delta_{\zeta\sigma}\ \delta_{ji}) \quad , \tag{43a}$$

and, again, note the cyclic form in $\underset{\approx}{\Lambda}_{b\sigma}^{cvk}$ with $\sigma \to b \to k \to v \to c$ ; see Figure 4. Substituting equation (41a) in equation (38b) yields

$$\hat{\underset{\approx}{h}}^{v} = \underset{\approx}{h}_{\infty}^{v}\ \underset{\approx}{v} + (\sum_{kb\sigma} \underset{\approx}{h}_{\infty}^{v}\ \underset{\approx}{y}_{k}\ \Lambda_{b\sigma}^{vk})\ \underset{\approx}{D}_{\sigma}^{b}\ \{\underset{\approx}{y}^{0b}\ \underset{\approx}{h}_{\infty}^{\sigma}\ \underset{\approx}{\sigma} + (\sum_{fe\xi} \underset{\approx}{\Lambda}_{e\xi}^{b\sigma f})\ \underset{\approx}{y}^{0e}\ \underset{\approx}{h}_{\infty}^{\xi}\ \underset{\approx}{\xi}\} \quad , \tag{44a}$$

where

$$\underset{\approx}{D}_{\sigma}^{b} = (\underset{\approx}{B}_{\sigma}^{b})^{-1} \quad . \tag{45a}$$

The reverberant terms in equation (44a) are clearly pure, and, therefore, so is $\hat{\underset{\approx}{h}}^{v}$. Thus, the construction of the $B_{\sigma}^{b}$'s , stated in equation (41a), achieved their designed purpose. The construction of the matrix $\underset{\approx}{B}_{\sigma}^{b}$ is central to the reverberant part of the response. Whenever a set of parameters in this matrix renders (the magnitude of) its eigenlike value small compared with unity, the reverberant response is high. A highly reverberant response is potentially indicated if the eigenlike values of $\underset{\approx}{B}_{\sigma}^{b}$ are small and numerous. Indeed the potential necessity for the existence of such small eigenlike values in $\underset{\approx}{B}_{\sigma}^{b}$, prompted its construction. In the frequency domain, a set of parameters that yield high reverberant response, as just prescribed, is dubbed a set of "resonance" parameters. A common example of a parameter in this set is the resonance frequency. It is

expected that invariably the contributing terms to the reverberation in the response will harbor the matrix factors $\underline{\underline{D}}_\sigma^b$ that are the inverse of the $\underline{\underline{D}}_\sigma^b$'s. [cf. equations (44a) and (45a).] It is noted that the matrix $\underline{\underline{B}}_\sigma^b$ is a temporal operator as $\underline{\underline{\Lambda}}_{e\xi}^{\sigma f}$ or, equivalently, $\underline{\underline{\Lambda}}_{e\xi}^{b\sigma f}$ is. In the spatial domain this operator is neuter. This statement is clarified noting that $\underline{\underline{y}}^{0b}\, \underline{\underline{h}}_\infty^\sigma\, \underline{\underline{y}}_f$ is not a spatial matrix operator. Since $\underline{\underline{B}}_\sigma^b$ is a temporal matrix operator so is, of course, $\underline{\underline{D}}_\sigma^b$. However, whereas $\underline{\underline{B}}_\sigma^b$ is a proper operator, its inverse $\underline{\underline{D}}_\sigma^b$ may need special attention when manipulated. Nonetheless, the expression for the impulse response matrix $\hat{\underline{\underline{h}}}^\nu$ is pure in the sense that it is a functional purely and solely of the properties of the dynamic systems and the couplings among them; namely, of the $\underline{\underline{h}}_\infty^\sigma$'s, $\hat{\underset{\sim}{\chi}}_a$'s, and $\underline{\underline{T}}_{b\sigma}^{\nu k}$'s. Again, making use of the simple model yields, for equations (41a) through (43a), the corresponding equations

$$\underline{\underline{B}}_\sigma^b\, \underline{\underline{\delta}}^{0b}\, \hat{\underline{\underline{h}}}^\sigma = \underline{\underline{\delta}}^{0b}\, \underline{\underline{h}}_\infty^\sigma\, \underline{\underline{\sigma}} + \Big(\sum_{fe\xi} \underline{\underline{\Lambda}}_{e\xi}^{b\sigma f}\Big)\, \underline{\underline{\delta}}^{0e}\, \underline{\underline{h}}_\infty^\xi\, \underline{\underline{\xi}} \quad , \tag{41b}$$

$$\underline{\underline{B}}_\sigma^b = \Big[\underline{\underline{I}} - \Big(\sum_{fe\xi} \underline{\underline{\Lambda}}_{e\xi}^{b\sigma f}\Big)\Big(\sum_{hc\zeta} \underline{\underline{\Lambda}}_{c\zeta}^{e\xi h}\Big) \sum_b \underline{\underline{\delta}}_{cb} \sum_\sigma \underline{\underline{\delta}}_{\zeta\sigma}\Big] \quad , \tag{42b}$$

$$\underline{\underline{\Lambda}}_{e\xi}^{b\sigma f} = \underline{\underline{\delta}}^{0b}\, \underline{\underline{h}}_\infty^\sigma\, \underline{\underline{\delta}}_f\, \underline{\underline{\Lambda}}_{e\xi}^{\sigma f} \; ; \quad \underline{\underline{\delta}}_{\zeta\sigma} = (\delta_{\zeta\sigma}\, \delta_{ji}) \quad , \tag{43b}$$

respectively. Substituting equation (41b) in equation (38c) one obtains for the simple model

$$\hat{\underline{\underline{h}}}^\nu = \underline{\underline{h}}_\infty^\nu\, \underline{\underline{\nu}} + \Big(\sum_{kb\sigma} \underline{\underline{h}}_\infty^\nu\, \underline{\underline{\delta}}_k\, \Lambda_{b\sigma}^{\nu k}\Big)\, \underline{\underline{D}}_\sigma^b\, \Big\{\underline{\underline{\delta}}^{0b}\, \underline{\underline{h}}_\infty^\sigma\, \underline{\underline{\sigma}}$$

$$+ \Big(\sum_{fe\xi} \underline{\underline{\Lambda}}_{e\xi}^{b\sigma f}\Big)\, \underline{\underline{\delta}}^{0e}\, \underline{\underline{h}}_\infty^\xi\, \underline{\underline{\xi}}\Big\} \; ; \qquad \underline{\underline{h}}_\infty^\nu = \underline{\underline{\bar{h}}}^\nu\, \underline{\underline{\Delta}}^\nu \quad , \tag{44b}$$

where

$$\underline{\underline{D}}_\sigma^b = (\underline{\underline{B}}_\sigma^b)^{-1} \quad . \tag{45b}$$

As in equation (45a), it is noted that the matrix $\underline{\underline{B}}_\sigma^b$ in equation (45b), is a temporal operator as $\underline{\underline{\Lambda}}_{e\xi}^{\sigma f}$

is. Like its counterpart in equation (45a), in the spatial domain this operator is neuter. This statement is clarified noting that $\underset{\approx}{\delta}^{0b}\ \underset{\approx}{h}^{\sigma}_{\infty}\ \underset{\approx}{\delta}_{f}$ is obviously not a spatial matrix operator.

Once equation (44) is derived, the grand equation for the response vector $\underset{\sim}{p}(\hat{\underset{\sim}{x}}, t)$ of the complex may be stated in terms of the impulse response matrix[1]

$$\hat{\underset{\approx}{h}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t') = \sum_{\nu} \hat{\underset{\approx}{h}}^{\nu}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t') \ ;$$

$$\hat{\underset{\approx}{h}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t') = \sum_{\nu}\sum_{\sigma} \{\underset{\approx}{h}^{\nu}_{\infty}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t')\, \underset{\approx}{\nu}\, \underset{\approx}{\delta}_{\nu\sigma} + \underset{\approx}{h}^{\nu}_{\sigma}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t')\, \underset{\approx}{\sigma}\} \ , \tag{46}$$

and the external drive vector[1]

$$\hat{\underset{\sim}{p}}_{e}(\hat{\underset{\sim}{x}}', t') = \sum_{\sigma} \underset{\approx}{\sigma}\ \underset{\sim}{p}_{e\sigma}(\hat{\underset{\sim}{x}}', t') \ . \tag{39}$$

in the form

$$[\hat{\underset{\approx}{h}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', t|t')\hat{\underset{\sim}{p}}_{e}(\hat{\underset{\sim}{x}}', t')] = \underset{\sim}{p}(\hat{\underset{\sim}{x}}, t) \ . \tag{47}$$

[cf. equations (8) through (10).] The expression for the impulse response matrix operator $\hat{\underset{\approx}{h}}$ in the second of equation (46) is of some interest. The double summation in $\hat{\underset{\approx}{h}}$ is introduced to accommodate the detail form of the drive $\hat{\underset{\sim}{p}}_{e} = \sum_{\sigma} \underset{\approx}{\sigma}\, \underset{\sim}{p}_{e\sigma}$. The first term in the curly brackets is the direct term and the second term describes the reverberant terms in the impulse response matrix operator. The direct term in the impulse response matrix operator is always readily distinguishable from the reverberant terms; equation (46) is a prime example of this statement. [cf. equation (38).]

Equation (47) is expressed in the temporal domain and is suitable, if not essential, to ascertaining the response of a complex to transient external drive systems - - external drives that are localized in the temporal domain. Inserting equation (44) in equation (47) results, even in models of simple structural complexes, in an equation that is cumbersome and difficult to interpret. Some relief may be derived if the complexes are assumed to be stationary in the temporal domain. This

kind of stationarity is imposed on the complex by assuming that the dynamic systems and the couplings among them remain unchanged with the passage of time; i.e.,

$$\underset{=}{h}^{v}_{\infty}(\underset{\sim}{\hat{x}}\,|\,\underset{\sim}{\hat{x}}', t\,|\,t') \equiv (2\pi)^{-1/2}\, \underset{=}{h}^{v}_{\infty}(\underset{\sim}{\hat{x}}\,|\,\underset{\sim}{\hat{x}}', t-t') \;;$$

$$\underset{=}{\Lambda}^{\sigma f}_{e\xi}(t''\,|\,t''') \equiv (2\pi)^{-1/2}\, \underset{=}{\Lambda}^{\sigma f}_{e\xi}(t''-t''') \;. \tag{48a}$$

[cf. equations (36a) and (38a).] In the simple model the conditions of stationarity are more stringently imposed; namely,

$$\underset{=}{h}^{v}_{\infty}(\underset{\sim}{\hat{x}}\,|\,\underset{\sim}{\hat{x}}', t\,|\,t') \equiv (2\pi)^{-1/2}\, \underset{=}{\bar{h}}^{v}_{\infty}(\underset{\sim}{\hat{x}}'\,|\,\underset{\sim}{\hat{x}}, t-t')\, \underset{=}{\Delta}(\underset{\sim}{\tau}^{v}) \;;$$

$$\underset{=}{\Lambda}^{vk}_{b\sigma} = \left(\Lambda^{vk}_{b\sigma ji}(0)\, \Delta\, (\tau^{vk}_{b\sigma ji})\, \delta^{0t}\right) \;, \tag{48b}$$

and the $\underset{=}{\Lambda}^{vk}_{b\sigma ji}(0)$'s and the $\underset{=}{\tau}^{vk}_{b\sigma ji}$'s are merely algebraic factors. [cf. equations (36b) and (38c).] Substituting equation (48) in equations (42), (44), and then in equation (46), helps, to be sure, but the formalism still remains cumbersome and difficult. Further relief may be derived if, in addition, the equation for the response is Fourier transformed from the temporal domain into the frequency domain. The equation in the Fourier domain is usually more suitable for ascertaining the response for "steady state" drives and for complexes that are stationary with respect to the temporal domain.

## IMPULSE RESPONSE OPERATORS IN THE FREQUENCY DOMAIN

To this point the analytical procedures and developments were conducted in the $\{\underset{\sim}{\hat{x}}, t\}$–space. Is there any advantage in transforming the equations derived in the preceding section into other related spaces; e.g., by a Fourier transformation? It was argued in the introduction that such a transformation, into a specified domain, may be advantageous when the complex is stationary with respect to the variables that define this domain. In this section, only a

temporal stationarity is considered. If the dynamic systems and the couplings among them remain unchanged with the passage of time, as defined and stated in equation (48), then under these conditions equation (30) may be recast, by a Fourier transformation, in the frequency domain

$$\underline{\underline{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', \omega)\, \underset{\sim}{p}_{ev}(\hat{\underset{\sim}{x}}', \omega) = \underset{\sim}{p}^{v}_{\infty}(\hat{\underset{\sim}{x}}, \omega) \quad , \tag{49a}$$

or in the abbreviated form

$$\underline{\underline{h}}^{v}_{\infty}(\omega)\, \underset{\sim}{p}_{ev}(\omega) = \underset{\sim}{p}^{v}_{\infty}(\omega) \quad , \tag{49b}$$

where

$$\underline{\underline{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', \omega) \rightarrow (2\pi)^{-1/2}\, \underline{\underline{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}} \,|\, \hat{\underset{\sim}{x}}', t-t')\, \exp[-i\omega(t-t')] \quad , \tag{50a}$$

$$\underset{\sim}{p}_{e\sigma}(\hat{\underset{\sim}{x}}, \omega) \rightarrow (2\pi)^{-1/2} \int dt'\, \underset{\sim}{p}_{e\sigma}(\hat{\underset{\sim}{x}}, t')\, \exp(-i\omega t') \quad , \tag{51a}$$

$$\underset{\sim}{p}^{v}_{\infty}(\hat{\underset{\sim}{x}}, \omega) \rightarrow (2\pi)^{-1/2} \int dt\, \underset{\sim}{p}^{v}_{\infty}(\hat{\underset{\sim}{x}}, t)\, \exp(-i\omega t) \quad . \tag{51b}$$

[cf. equation (1).] It is also noted that the arrows, replacing the equality signs, indicate that the quantities are related by Fourier transformations. However, unlike the notational procedure adapted in equations (1) through (3), the letterings remain the same. This, together with the reduction of equation (49a) to equation (49b), is instituted to keep the notations, which are already unwieldly, to a minimum. The relief in transforming from the time domain into the frequency domain is achieved mainly because the temporal operators in the temporally stationary formalism are eigen-operators with respect to the Fourier eigenfunction $(2\pi)^{-1/2} \exp(i\omega t)$. Therefore, in the frequency domain these temporal operators are simply replaced by their eigen-values. These eigen-values are merely algebraic factors. This replacement is particularly explicit in the formalism of the simple model. For the simple model, equation (50a) can be cast in the form

$$\underset{\approx}{h}^{v}_{\infty}(\hat{\underset{\sim}{x}}\,|\,\hat{\underset{\sim}{x}}',\omega) \rightarrow \bar{\underset{\approx}{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}}\,|\,\hat{\underset{\sim}{x}}',t-t')\,\underset{\approx}{\Delta}(\underset{\sim}{\tau}^{v})\,(2\pi)^{-1/2}\exp[-i\omega(t-t')] \;;$$

$$\underset{\approx}{h}^{v}_{\infty}(\hat{\underset{\sim}{x}}\,|\,\hat{\underset{\sim}{x}}',\omega) = \bar{\underset{\approx}{h}}^{v}_{\infty}(\hat{\underset{\sim}{x}}\,|\,\hat{\underset{\sim}{x}}',\omega)\,\underset{\approx}{\nabla}(\omega\underset{\sim}{\tau}^{v}) \equiv \underset{\approx}{h}^{v}_{\infty}(\omega) \;;$$

$$\left.\begin{matrix}\Delta(\tau^{v}_{jj})\\ F[\Delta(\tau^{v}_{jj})]\end{matrix}\right\}[(2\pi)^{-1/2}\exp(i\omega t)] = \left.\begin{matrix}\nabla(\omega\tau^{v}_{jj})\\ F[\nabla(\omega\tau^{v}_{jj})]\end{matrix}\right\}[(2\pi)^{-1/2}\exp(i\omega t)] \;;$$

$$\underset{\approx}{\nabla}(\omega\underset{\sim}{\tau}^{v}) = \big(\exp(-i\omega\tau^{v}_{jj})\delta_{ji}\big) = \big(\nabla(\omega\tau^{v}_{jj})\delta_{ji}\big) \;. \tag{50b}$$

where F is an arbitrary, but well behaved, functional of the dependent quantity. Imposing the temporal stationarity, stated in equation (48), on equations (42) and (43), one obtains

$$\underset{\approx}{B}^{b}_{\sigma}(\omega) = [\,\underset{\approx}{I} - (\sum_{fe\xi}\underset{\approx}{\Lambda}^{b\sigma f}_{e\xi}(\omega))\,(\sum_{hc\xi}\underset{\approx}{\Lambda}^{e\xi h}_{c\zeta}(\omega))\sum_{b}\underset{\approx}{\delta}_{cb}\sum_{\sigma}\underset{\approx}{\delta}_{\zeta\sigma}\,] \;, \tag{52}$$

where for the more general model

$$\underset{\approx}{\Lambda}^{b\sigma f}_{e\xi}(\omega) = \underset{\approx}{h}^{\sigma}_{\infty bf}(\omega)\,\underset{\approx}{\Lambda}^{\sigma f}_{e\xi}(\omega) \;;$$

$$\underset{\approx}{\Lambda}^{\sigma f}_{e\xi}(\omega) \rightarrow (2\pi)^{-1/2}\,\underset{\approx}{\Lambda}^{\sigma f}_{e\xi}(t''-t''')\,\exp[-i\omega(t''-t''')] \;;$$

$$\underset{\approx}{h}^{\sigma}_{\infty bf} = \underset{\approx}{y}^{0b}\,\underset{\approx}{h}^{\sigma}_{\infty}(\omega)\,\underset{\approx}{y}_{f} \;, \tag{53a}$$

and for the simple model

$$\underset{\approx}{\Lambda}_{e\xi}^{b\sigma f}(\omega) = \underset{\approx}{h}_{\infty bf}^{\sigma}(\omega)\ \underset{\approx}{\Lambda}_{e\xi}^{\sigma f}(\omega)\ ;$$

$$\underset{\approx}{\Lambda}_{e\xi}^{\sigma f}(\omega) = \left(\Lambda_{e\xi ji}^{\sigma f}(0)\,\nabla(\omega\tau_{e\xi ji}^{\sigma f})\right)\ ; \quad \nabla(\omega\tau_{e\xi ji}^{\sigma f}) = \exp(-i\,\omega\tau_{e\xi ji}^{\sigma f})\ ;$$

$$\underset{\approx}{h}_{\infty bf}^{\sigma}(\omega) = \underset{\approx}{\bar{h}}_{\infty bf}^{\sigma}(\omega)\ \underset{\approx}{\nabla}(\omega\underset{\sim}{\tau}_{bf}^{\sigma})\ ; \quad \underset{\approx}{\bar{h}}_{\infty bf}^{\sigma}(\omega) = \underset{\approx}{\delta}^{0b}\ \underset{\approx}{\bar{h}}_{\infty}^{\sigma}(\omega)\ \underset{\approx}{\delta}_{f}\ ;$$

$$\underset{\approx}{\nabla}(\omega\underset{\sim}{\tau}_{bf}^{\sigma}) = \left(\exp(-i\,\omega\tau_{bfj}^{\sigma})\ \delta_{ji}\right) = \left(\nabla(\omega\tau_{bfj}^{\sigma})\,\delta_{ji}\right)\ . \tag{53b}$$

[cf. equation (40).] As expected and anticipated, equation (53b) is more explicit and simpler than equation (53a). Imposing the temporal stationarity, stated in equation (48), on equations (44a) and (44b) yields

$$\underset{\approx}{\hat{h}}^{v}(\omega) = \underset{\approx}{h}_{\infty}^{v}(\omega)\ \underset{\approx}{v} + \sum_{kb\sigma}\ \underset{\approx}{h}_{\infty}^{v}(\omega)\ \underset{\approx}{y}_{k}\ \underset{\approx}{\Lambda}_{b\sigma}^{vk}(\omega)\ \underset{\approx}{D}_{\sigma}^{b}(\omega)$$

$$\{\underset{\approx}{y}^{0b}\ \underset{\approx}{h}_{\infty}^{\sigma}(\omega)\ \underset{\approx}{\sigma} + \sum_{fe\xi}\ \underset{\approx}{\Lambda}_{e\xi}^{b\sigma f}(\omega)\ \underset{\approx}{y}^{0e}\ \underset{\approx}{h}_{\infty}^{\xi}(\omega)\ \underset{\approx}{\xi}\ \}\ , \tag{54a}$$

$$\underset{\approx}{\hat{h}}^{v}(\omega) = \underset{\approx}{h}_{\infty}^{v}(\omega)\ \underset{\approx}{v} + \sum_{kb\sigma}\ \underset{\approx}{h}_{\infty}^{v}(\omega)\ \underset{\approx}{\delta}_{k}\ \underset{\approx}{\Lambda}_{b\sigma}^{vk}(\omega)\ \underset{\approx}{D}_{\sigma}^{b}(\omega)$$

$$\{\underset{\approx}{\delta}^{0b}\ \underset{\approx}{h}_{\infty}^{\sigma}(\omega)\ \underset{\approx}{\sigma} + \sum_{fe\xi}\ \underset{\approx}{\Lambda}_{e\xi}^{b\sigma f}(\omega)\ \underset{\approx}{\delta}^{0e}\ \underset{\approx}{h}_{\infty}^{\xi}(\omega)\ \underset{\approx}{\xi}\}\ , \tag{54b}$$

respectively, where

$$\underset{\approx}{D}_{\sigma}^{b}(\omega) = [\underset{\approx}{B}_{\sigma}^{b}(\omega)]^{-1}\ . \tag{55}$$

[cf. equation (45).] It is noted that the matrix $\underset{\approx}{B}_{\sigma}^{b}(\omega)$, unlike $\underset{\approx}{B}_{\sigma}^{b}$ in equation (42), is merely an algebraic matrix factor, and therefore so is $D_{\sigma}^{b}(\omega)$. Consequently, in the frequency domain, the

impulse response matrix $\hat{\underline{\underline{h}}}(\omega)$ is of simpler structure than it is in the temporal domain. Again, the direct term $\underline{\underline{h}}^{\nu}_{\infty}\,\underline{\underline{\nu}}$ in the impulse response matrix operator, in equation (53 a&b), is clearly distinguishable from the reverberant terms. Moreover, it emerges that the grand equation for the response vector $\underset{\sim}{p}(\hat{\underset{\sim}{x}}, \omega)$ of a model that is temporally stationary is, as expected, algebraic in the frequency domain

$$[\hat{\underline{\underline{h}}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega)\ \hat{\underset{\sim}{p}}_e(\hat{\underset{\sim}{x}}', \omega)] = \underset{\sim}{p}(\hat{\underset{\sim}{x}}, \omega) \quad , \tag{56}$$

where

$$\hat{\underline{\underline{h}}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega) = \sum_{\nu} \hat{\underline{\underline{h}}}^{\nu}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega) \quad ;$$

$$\hat{\underline{\underline{h}}}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega) = \sum_{\nu}\sum_{\sigma} \{\underline{\underline{h}}^{\nu}_{\infty}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega)\,\underline{\underline{\nu}}\ \underline{\underline{\delta}}_{\nu\sigma} + \underline{\underline{h}}^{\nu}_{\sigma}(\hat{\underset{\sim}{x}}|\hat{\underset{\sim}{x}}', \omega)\,\underline{\underline{\sigma}}\} \quad , \tag{57}$$

$$\hat{p}_e(\hat{\underset{\sim}{x}}', \omega) = \sum_{\sigma} \underline{\underline{\sigma}}\, p_{e\sigma}(\hat{\underset{\sim}{x}}', \omega) \quad , \tag{58}$$

are the impulse response matrix and the external drive vector, respectively, expressed in the frequency domain. [cf. equations (39), (46), and (47).]

The relief, expected from the stationarity of the complex in the temporal domain and the Fourier transformation of equation (47), is materialized in equation (56). Nonetheless, as indicated by equations (53) and (54), substantial cumbersomeness and difficulty still remain in equation (56). This stems, as it does in equation (47), from the multi-spatial dimensionality of the complex. How much relief may then be reasonably gained by restricting the model of the complex to a composition of coupled one-dimensional dynamic systems?

## COUPLED ONE-DIMENSIONAL DYNAMIC SYSTEMS

As argued in the introduction, if an impulse response operator is stationary in a given dependent variable, it can be rendered, by a Fourier transformation, algebraic in this variable. Then, by focusing on a single value of the Fourier conjugate of the variable, the equation for the response is effectively reduced in the dimensionality of this variable [3,4]. Thus, if the complex is stationary with respect to $\hat{x}_2$ in the spatial domain $\hat{x} = \{x, \hat{x}_2\}$, the equation for the response can be reduced, as just specified, to account for a model of a complex composed of spatially one-dimensional dynamic systems, where conveniently $x_1 \equiv x$, and the reduced equation is specified at a specific value of $\hat{k}_2$; $\hat{k}_2$ being the Fourier conjugate of $\hat{x}_2$. In this paper, for the sake of brevity, this one-dimensionality of the dynamic systems is imposed directly on equations (42) through (44) and on equations (51) through (53). The reference to the other spatial dimensionalities, if present, is simply suppressed and omitted in the Fourier transform. The authors previously presented, in References 4 and 5, an analogeous analysis of the response of a model composed of multiple coupled one-dimensional dynamic systems. In these previous considerations, however, the one-dimensionality of the dynamic systems is imposed a priori. In the first reference, the analysis is conducted a priori in the $\{x, \omega\}$–space. In the second, and the more recent reference, the analysis is conducted a priori in the $\{x, t\}$–space. In this way the relationship between subsequent considerations, in this section, and the previous considerations, in References 4 and 5, can be established with the insights that such a comparison may provide to the modeling and the formalism.

A complex consisting of one-dimensional dynamic systems is sketched in Figure 5. The complex is appropriately defined in terms of just two junctions, junction r and junction q [1-5]. Thus, the model is specified in terms of two propagator matrices, two terminal vectors, and two junction matrices. For the more general model, the propagator matrices are in the forms

$$\underset{\approx}{h}_{\infty}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t') = \left(h_{\infty}^{\alpha}(x_j\,|\,x_j',t\,|\,t')\,\delta_{ji}\right) \;\; ; \;\; \underset{\sim}{x} = \{x_j\} \;\; ;$$

$$x_s \underset{s=1}{\longrightarrow} x \;\; ; \; x_{1j} \longrightarrow x_j \; ; \;\; \alpha = r \text{ or } q \;\; , \tag{59a}$$

and for the simple model in the forms

$$\underset{\approx}{h}_{\infty}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t') = \underset{\approx}{\bar{h}}_{\infty}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t')\;\underset{\approx}{\Delta}(\underset{\sim}{\tau}^{\alpha}) \;\; ;$$

$$\underset{\approx}{\bar{h}}_{\infty}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t') = \left(\underset{\approx}{\bar{h}}_{\infty}^{\alpha}(x_j\,|\,x_j',t\,|\,t')\delta_{ji}\right) \;\; ;$$

$$\underset{\sim}{\tau}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}') = \{\tau_j^{\alpha}(x_j\,|\,x_j')\} = \{\tau_{jj}^{\alpha}\} \;\; . \tag{59b}$$

The two terminal position vectors are specified in the forms

$$\underset{\sim}{x}_{\alpha} = \{x_{\alpha j}\} \;\; ; \;\; \alpha = r \text{ or } q \;\; , \tag{60}$$

These terminal position vectors are common to the more general and the simple models. The two junction matrices for the more general model assume the forms

$$\underset{\approx}{T}_{b\sigma}^{vk} = \underset{\approx}{\delta}_{bk}\,\underset{\approx}{\delta}_{b\sigma}\,\underset{\approx}{\delta}_{\sigma\alpha}\,(\underset{\approx}{I} - \underset{\approx}{\delta}_{v\sigma})\;\underset{\approx}{T}_{\alpha} \;\; ;$$

$$\underset{\approx}{T}_{\alpha} = \underset{\approx}{\delta}\,(\underset{\sim}{x}'' - \underset{\sim}{x}_{\alpha})\,\underset{\approx}{\Lambda}_{\alpha} \int \underset{\approx}{\delta}\,(\underset{\sim}{x}_{\alpha} - \underset{\sim}{x}''')\,d\,\underset{\approx}{x}'''\cdots \;\; ;$$

$$\underset{\approx}{\Lambda}_{\alpha} = \left(\Lambda_{\alpha ji}\right) \;\; ; \;\; \alpha = r \text{ or } q \;\; ;$$

$$\underset{\approx}{\Lambda}_{\alpha}\,(t''\,|\,t''')\,f_e(t''') = \underset{\approx}{\Lambda}_{\alpha}\,(0)\,f\,(t'') \;\; , \tag{61a}$$

and for the simple model they assume the forms

$$\underline{\underline{T}}^{vk}_{b\sigma} = \underline{\underline{\delta}}_{bk}\, \underline{\underline{\delta}}_{b\sigma}\, \underline{\underline{\delta}}_{\sigma\alpha} (\underline{\underline{I}} - \underline{\underline{\delta}}_{v\sigma})\, \underline{\underline{T}}_{\alpha} \;;$$

$$\underline{\underline{T}}_{\alpha} = \underline{\underline{\delta}}\,(\underset{\sim}{x} - \underset{\sim}{x}_{\alpha})\, \underline{\underline{\Lambda}}_{\alpha} \int \underline{\underline{\delta}}\,(\underset{\sim}{\hat{x}}_{\alpha} - \underset{\sim}{\hat{x}}')\, d\,\underline{\underline{x}}' \cdots \;;$$

$$\underline{\underline{\Lambda}}_{\alpha} = \left(\Lambda_{\alpha ji}\,(\tau_{\alpha ji})\, \delta^{0\iota}\right) \;; \qquad \Lambda_{\alpha ji}\,(\tau_{\alpha ji}) = \Lambda_{\alpha ji}\,(0)\, \Delta\,(\tau_{\alpha ji}) \;. \tag{61b}$$

[cf. equations (30) through (36).] Substituting equation (59) and (61) in equations (42), (46), and (44) yields

$$\underline{\underline{B}}_{\alpha} = [\underline{\underline{I}} - \underline{\underline{\Lambda}}^{\alpha}_{\beta}\, \underline{\underline{\Lambda}}^{\beta}_{\alpha}] \;; \quad \underline{\underline{D}}_{\alpha} = (\underline{\underline{B}}_{\alpha})^{-1} \;;$$

$$\underline{\underline{\Lambda}}^{\beta}_{\alpha} = \underline{\underline{\delta}}^{0\beta}\, \underline{\underline{h}}^{\beta}_{\infty}\, \underline{\underline{\delta}}_{\alpha}\, \underline{\underline{\Lambda}}_{\alpha} \;, \tag{62}$$

$$\underline{\underline{\hat{h}}} = \sum_{r,q} \underline{\underline{\hat{h}}}^{\alpha} \;; \quad \underline{\underline{\hat{h}}}^{\alpha} = (\underline{\underline{h}}^{\alpha}_{\infty} + \underline{\underline{h}}^{\alpha}_{\alpha})\, \underline{\underline{\alpha}} + \underline{\underline{h}}^{\alpha}_{\beta}\, \underline{\underline{\beta}} \;, \tag{63}$$

$$\underline{\underline{h}}^{\alpha}_{\alpha} = \underline{\underline{h}}^{\alpha}_{\infty}\, \underline{\underline{\delta}}_{\beta}\, \underline{\underline{\Lambda}}_{\beta}\, \underline{\underline{D}}_{\beta}\, \underline{\underline{\Lambda}}^{\beta}_{\alpha}\, \underline{\underline{\delta}}^{0\alpha}\, \underline{\underline{h}}^{\alpha}_{\infty} \;;$$

$$\underline{\underline{h}}^{\alpha}_{\beta} = \underline{\underline{h}}^{\alpha}_{\infty}\, \underline{\underline{\delta}}_{\beta}\, \underline{\underline{\Lambda}}_{\beta}\, \underline{\underline{D}}_{\beta}\, \underline{\underline{\delta}}^{0\beta}\, \underline{\underline{h}}^{\beta}_{\infty} \;, \tag{64}$$

respectively. It is noted that the matrix quantities $\underline{\underline{\Lambda}}_{\alpha}$ and $\underline{\underline{\Lambda}}^{\beta}_{\alpha}$ are temporal operators and, therefore, so is $\underline{\underline{B}}_{\alpha}$ and, consequently, so is $\underline{\underline{D}}_{\alpha}$. In this sense equations (62) and (64) cannot be manipulated cavalierly; commutations and expansion procedures must be carefully viewed and performed. Equations (59), (61), and (62) through (64) are the equations for the response of the model of a complex composed of one-dimensional dynamic systems. Compared with equations (35), (36), and (41) through (44) for the multi-dimensional dynamic systems, the reduction achieved in equations (59), and (61) through (64) is dramatic. Moreover, it is noted that the spatial operative properties of the junction matrices in equations (59a) and (61a) and those in equations (59b) and (61b) are identical; a further and considerable simplification indeed. The difference in the junction matrices described in these two sets of equations lies entirely in the temporal domain, but the one in

equation (59b) and (61b) are by far the simpler. Finally, it is observed, from equations (63) and (64), that the impulse response matrix operator $\underset{\approx}{\hat{h}}$, for the model composed of coupled one-dimensional dynamic systems, harbors six (6) distinct terms: two (2) direct terms; $\underset{\approx}{h}{}^{\alpha}_{\infty}\ \underset{\approx}{\alpha}$, $\alpha$ = r and q, and four (4) reverberant terms; $\underset{\approx}{h}{}^{\alpha}_{\alpha}\ \underset{\approx}{\alpha}$ and $\underset{\approx}{h}{}^{\alpha}_{\beta}\ \underset{\approx}{\beta}$, $\alpha$ = r and q. Although the impulse response matrix operator for the one-dimensional model is significantly simpler, some further simplification in the model is still required to attain reasonable descriptions. The first in this process is to bring to bear the temporal stationarity onto the one-dimensional model. Thus, imposing the stationarity in the temporal domain on the one-dimensional model recasts equation (48a) in the form

$$\underset{\approx}{h}{}^{\alpha}_{\infty}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t') \equiv (2\pi)^{-1/2}\,\underset{\approx}{h}{}^{\alpha}_{\infty}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t-t') \quad ;$$

$$\underset{\approx}{\Lambda}_{\alpha}(t''\,|\,t''') \equiv (2\pi)^{-1/2}\,\underset{\approx}{\Lambda}_{\alpha}\,(t''-t''') \quad , \tag{65a}$$

and for the simple model, equation (48b) yields

$$\underset{\approx}{h}{}^{\alpha}_{\infty}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t\,|\,t') \equiv (2\pi)^{-1/2}\ \underset{\approx}{\bar{h}}{}^{\alpha}_{\infty}(\underset{\sim}{x}'\,|\,\underset{\sim}{x},t-t')\ \underset{\approx}{\Delta}(\underset{\sim}{\tau}^{\alpha}) \quad ;$$

$$\underset{\approx}{\Lambda}_{\alpha} = \left(\Lambda_{\alpha ji}(0)\ \ \Delta(\tau_{\alpha ji})\ \delta^{01}\right) \quad , \tag{65b}$$

and the $\underset{\approx}{\Lambda}_{\alpha}$ (0)'s and the $\underset{\approx}{\tau}_{\alpha}$'s are algebraic factors. Substituting equations (65a) and (65b) in equations (62) through (64) yields further simplification. However, the significant simplification is attained when the resulting equations are transformed into the frequency domain. Such Fourier transformation yields

$$\underset{\approx}{B}_{\alpha}(\omega) = [\underset{\approx}{I} - \underset{\approx}{\Lambda}{}^{\alpha}_{\beta}(\omega)\ \ \underset{\approx}{\Lambda}{}^{\beta}_{\alpha}(\omega)] \quad ; \qquad \underset{\approx}{D}_{\alpha}(\omega) = [\underset{\approx}{B}_{\alpha}(\omega)]^{-1} \quad ;$$

$$\underset{\approx}{\Lambda}{}^{\beta}_{\alpha}(\omega) = \underset{\approx}{\delta}^{0\beta}\ \underset{\approx}{h}{}^{\beta}_{\infty}(\omega)\ \underset{\approx}{\delta}_{\alpha}\ \underset{\approx}{\Lambda}_{\alpha}(\omega) = \underset{\approx}{g}{}^{\beta}_{\infty}(\underset{\sim}{x}_{\beta}\,|\,\underset{\sim}{x}_{\alpha},\omega)\ \underset{\approx}{\Lambda}_{\alpha}(\omega) \quad , \tag{66}$$

$$\hat{\underset{\approx}{h}}(\omega) = \sum_{r,q} \hat{\underset{\approx}{h}}^{\alpha}(\underset{\approx}{q}) \; ; \qquad \hat{\underset{\approx}{h}}^{\alpha}(\omega) = [\underset{\approx}{h}^{\alpha}_{\infty}(\omega) + \underset{\approx}{h}^{\alpha}_{\alpha}(\omega)] \; \underset{\approx}{\alpha} + \underset{\approx}{h}^{\alpha}_{\beta}(\omega) \; \underset{\approx}{\beta} \quad , \tag{67}$$

$$\underset{\approx}{h}^{\alpha}_{\alpha}(\omega) = \underset{\approx}{h}^{\alpha}_{\infty}(\omega) \; \underset{\approx}{\delta}_{\beta} \; \underset{\approx}{\Lambda}_{\beta}(\omega) \; \underset{\approx}{D}_{\beta}(\omega) \; \underset{\approx}{\Lambda}^{\beta}_{\alpha}(\omega) \; \underset{\approx}{\delta}^{0\alpha} \; \underset{\approx}{h}^{\alpha}_{\infty}(\omega) \; ;$$

$$\underset{\approx}{h}^{\alpha}_{\beta}(\omega) = \underset{\approx}{h}^{\alpha}_{\infty}(\omega) \; \underset{\approx}{\delta}_{\beta} \; \underset{\approx}{\Lambda}_{\beta}(\omega) \; \underset{\approx}{D}_{\beta}(\omega) \; \underset{\approx}{\delta}^{0\beta} \; \underset{\approx}{h}^{\beta}_{\infty}(\omega) \quad , \tag{68}$$

respectively, where for the general model

$$\underset{\approx}{g}^{\beta}(\underset{\sim}{x}_{\beta} | \underset{\sim}{x}_{\alpha}, \omega) \rightarrow (2\pi)^{-1/2} \; \underset{\approx}{\delta}^{0\beta} \; \underset{\approx}{h}^{\beta}(\underset{\sim}{x} | \underset{\sim}{x}', t-t') \; \underset{\approx}{\delta}_{\alpha} \; \exp[-i\omega(t-t')] \quad ;$$

$$\underset{\approx}{\Lambda}_{\alpha}(\omega) \rightarrow (2\pi)^{-1/2} \, \underset{\approx}{\Lambda}_{\alpha}(t'' - t''') \; \exp[-i\omega(t'' - t''')] \quad . \tag{69a}$$

and for the simple model

$$\underset{\approx}{g}^{\beta}_{\infty}(\underset{\sim}{x}_{\beta} | \underset{\sim}{x}_{\alpha}, \omega) = \underset{\approx}{\bar{g}}^{\beta}_{\infty}(\underset{\sim}{x}_{\beta} | \underset{\sim}{x}_{\alpha}, \omega) \; \underset{\approx}{\nabla}(\omega \underset{\sim}{\tau}^{\beta}_{\beta\alpha}) \; ;$$

$$\underset{\approx}{\bar{g}}^{\beta}_{\infty}(\underset{\sim}{x}_{\beta} | \underset{\sim}{x}_{\alpha}, \omega) \rightarrow (2\pi)^{-1/2} \; \underset{\approx}{\delta}^{0\beta} \; \underset{\approx}{\bar{h}}^{\beta}_{\infty}(\underset{\sim}{x} | \underset{\sim}{x}', t-t') \; \underset{\approx}{\delta}_{\alpha} \exp[-i\omega(t-t')] \quad ;$$

$$\underset{\approx}{\nabla}(\omega \underset{\sim}{\tau}^{\beta}_{\beta\alpha}) = \left(\exp[-i\omega \; \tau^{\beta}_{j}(x_{\beta j} | x_{\alpha j})] \; \delta_{ji}\right) \quad ;$$

$$\underset{\approx}{\Lambda}_{\alpha}(\omega) = \left(\Lambda_{\alpha ji}(0) \, \nabla(\omega \tau_{\alpha ji})\right) \quad ; \qquad \nabla(\omega \tau_{\alpha ji}) = \exp(-i\omega \; \tau_{\alpha ji}) \quad . \tag{69b}$$

where $\underset{\approx}{g}^{\alpha}_{\infty}(\omega)$ and $\underset{\approx}{\bar{g}}^{\alpha}_{\infty}(\omega)$ are the impulse response functions that are associated with the impulse response operators $\underset{\approx}{h}^{\alpha}_{\infty}(\omega)$ and $\underset{\approx}{\bar{h}}^{\alpha}_{\infty}(\omega)$, respectively. [cf. equations (4) and (40).] Comparing equations (66) through (69) with equations (50) through (54), again, reveals the considerable simplification attained in modeling the complex in terms of coupled one-dimensional dynamic systems. However, to make ready contact with the formalisms developed and used in Reference 5 and References 1 and 2, further simplifying assumptions need to be imposed on the one-dimensional model of the complex.

## BASIC COUPLED ONE-DIMENSIONAL DYNAMIC SYSTEMS

Using equations (65b) and assuming the coupled one-dimensional dynamic systems to be basic, one may explicitly define and impose

$$
\begin{aligned}
&(2\pi)^{-1/2}\;\bar{\underset{\approx}{h}}{}^{\alpha}_{\infty}(\underset{\sim}{x}\,|\,\underset{\sim}{x}',t-t') = \bar{\bar{\underset{\approx}{h}}}{}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}')\;\delta^{0t}(t-t')\;\;; \\
&\underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}^{\alpha}) = \underset{\approx}{U}^{\alpha+}(\underset{\sim}{x}-\underset{\sim}{x}')\;\;\underset{\approx}{\Delta}(\underset{\sim}{\tau}^{\alpha})\;\;;\qquad \bar{\bar{\underset{\approx}{g}}}{}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}') \equiv \underset{\approx}{U}^{\alpha+}(\underset{\sim}{x}-\underset{\sim}{x}')\;\;; \\
&\underset{\sim}{\tau}^{\alpha} = \{(C_j^{\alpha})^{-1}\,|\,x_j - x_j'\,|\,\}\;\;;\qquad C_j^{\alpha} = C_{0j}^{\alpha}\,(1-i\eta_j^{\alpha})^{-1}\;\;; \\
&\underset{\approx}{U}^{\alpha+}(\underset{\sim}{x}-\underset{\sim}{x}') = \left(U_j^{\alpha+}(x_j-x_j')\,\delta_{ji}\right)\;\;;\qquad U_j^{\alpha+}(x_j-x_j') = U[(x_j-x_j')\,\mathrm{sign}\,(x_{\alpha j}-x_{\beta j})]\;\;; \\
&\qquad\qquad\alpha = r \text{ or } q\;\;;\quad \beta = r \text{ or } q\;\;;\quad \alpha \neq \beta\;,
\end{aligned}
\tag{70}
$$

where $C_{0j}^{\alpha}$ is a temporally independent speed of propagation toward junction $\alpha$ and $\eta_j^{\alpha}$ is the loss factor associated with this propagation in the (j)th dynamic system [3]. [cf. equations (6b) and (59).] Consulting equation (41), and using equations (62), (65b), and (70), one obtains

$$
\begin{aligned}
&\underset{\approx}{B}_{\alpha} = [\underset{\approx}{I} - \underset{\approx}{\Lambda}{}_{\beta}^{\alpha}\,\underset{\approx}{\Lambda}{}_{\alpha}^{\beta}]\;;\quad \underset{\approx}{D}_{\alpha} = (\underset{\approx}{B}_{\alpha})^{-1}\;\;;\quad \underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}{}_{\alpha\beta}^{\alpha}) \equiv \underset{\approx}{\Delta}(\underset{\sim}{\tau}{}_{\alpha\beta}^{\alpha})\;\;; \\
&\underset{\approx}{\Lambda}{}_{\alpha}^{\beta} = \underset{\approx}{\Delta}(\underset{\sim}{\tau}{}_{\beta\alpha}^{\beta})\;\underset{\approx}{\Lambda}_{\alpha}\;\;;\quad \underset{\sim}{\tau}{}_{\beta\alpha}^{\beta} = \{(C_j^{\alpha})^{-1}\,|\,x_{\beta j} - x_{\alpha j}\,|\}\;\;.
\end{aligned}
\tag{71}
$$

Moreover, from equations (63), (64), (65b), and (70) one obtains

$$
\hat{\underset{\approx}{h}}{}^{\alpha} = \int\left\{[\underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}^{\alpha}) + \underset{\approx}{g}{}_{0\alpha}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}')]\,\underset{\approx}{\alpha} + \underset{\approx}{g}{}_{0\beta}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}')\,\underset{\approx}{\beta}\right\}\,d\underset{\approx}{x}'\;\;\delta^{0t}(t-t')\cdots\;\;,
\tag{72}
$$

$$
\begin{aligned}
&\underset{\approx}{g}{}_{0\alpha}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}') = \underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}{}_{/\beta}^{\alpha})\;\underset{\approx}{\Lambda}_{\beta}\;\underset{\approx}{D}_{\beta}\;\;\underset{\approx}{\Delta}^{\beta}(\underset{\sim}{\tau}{}_{\beta\alpha}^{\beta})\;\underset{\approx}{\Lambda}_{\alpha}\;\;\underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}{}_{\alpha/}^{\alpha})\;\;; \\
&\underset{\approx}{g}{}_{0\beta}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}') = \underset{\approx}{\Delta}^{\alpha+}(\underset{\sim}{\tau}{}_{/\beta}^{\alpha})\;\underset{\approx}{\Lambda}_{\beta}\;\underset{\approx}{D}_{\beta}\;\;\underset{\approx}{\Delta}^{\beta+}(\underset{\sim}{\tau}{}_{\beta/}^{\alpha})\;\;,
\end{aligned}
\tag{73}
$$

where $\underline{\underline{D}}_{\beta}$, in equation (73), is as stated in equation (71), and $\underset{\sim}{\tau}^{\alpha}_{\alpha/}$, $\underset{\sim}{\tau}^{\alpha}_{/\beta}$, and $\underset{\sim}{\tau}^{\alpha}_{\alpha\beta}$, are deduced from equation (40). Equations (39), (47), (63), and (71) through (73) are employed in Reference 5 to exemplify the nature of the response vector $\underset{\sim}{p}\,(\underset{\sim}{x}, t)$ of a model of a complex composed of multiple coupled basic one-dimensional dynamic systems. In this reference, however, the one-dimensionality of the basic dynamic systems was assumed a priori.

Using equations (69b) and assuming the coupled one-dimensional dynamic systems to be basic, one may explicitly define

$$\bar{\underline{\underline{h}}}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}', \omega) = \bar{\underline{\underline{h}}}^{\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}') \quad ; \quad \underline{\underline{\nabla}}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}) = \underline{\underline{U}}^{\alpha+}(\underset{\sim}{x}-\underset{\sim}{x}')\ \underline{\underline{\nabla}}(\omega\underset{\sim}{\tau}^{\alpha}) \ ;$$

$$\underset{\sim}{\tau}^{\alpha} = \{(C_j^{\alpha})^{-1}\,|x_j - x_j'|\} \ . \tag{74}$$

[cf. equation (6b).] From equations (66) and (69b ) one obtains

$$\underline{\underline{B}}_{\alpha}(\omega) = [\underline{\underline{I}} - \underline{\underline{\Lambda}}^{\alpha}_{\beta}(\omega)\ \underline{\underline{\Lambda}}^{\beta}_{\alpha}(\omega)] \ ; \quad \underline{\underline{D}}_{\alpha}(\omega) = [\underline{\underline{B}}_{\alpha}(\omega)]^{-1} \ ;$$

$$\underline{\underline{\Lambda}}^{\beta}_{\alpha}(\omega) = \underline{\underline{\nabla}}^{\beta}_{\beta\alpha}\ \underline{\underline{\Lambda}}_{\alpha}(\omega) \ ; \quad \underline{\underline{\nabla}}^{\beta}_{\beta\alpha} = \left(\exp(-i\omega\tau^{\beta}_{\beta\alpha j})\,\delta_{ji}\right) = \left(\nabla(\tau^{\beta}_{\beta\alpha j})\delta_{ji}\right) . \tag{75}$$

[cf. equations (40) and (71).] Moreover, from equations (67), (68), (69b), and (74) one obtains

$$\hat{\underline{\underline{h}}}^{\alpha}(\omega) = \int \left\{[\underline{\underline{\nabla}}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}) + \underline{\underline{g}}^{\alpha}_{0\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}', \omega)]\,\underline{\underline{\alpha}} + \underline{\underline{g}}^{\alpha}_{0\beta}(\underset{\sim}{x}\,|\,\underset{\sim}{x}', \omega)\,\underline{\underline{\beta}}\right\} d\underset{\sim}{x}' \cdots \ , \tag{76}$$

$$\underline{\underline{g}}^{\alpha}_{0\alpha}(\underset{\sim}{x}\,|\,\underset{\sim}{x}', \omega) = \underline{\underline{\nabla}}^{\alpha+}_{/\beta}\ \underline{\underline{\Lambda}}_{\beta}(\omega)\ \underline{\underline{D}}_{\beta}(\omega)\ \underline{\underline{\nabla}}^{\beta}_{\beta\alpha}\ \underline{\underline{\Lambda}}_{\alpha}(\omega)\ \underline{\underline{\nabla}}^{\alpha+}_{\alpha/} \ ;$$

$$\underline{\underline{g}}^{\alpha}_{0\beta}(\underset{\sim}{x}\,|\,\underset{\sim}{x}', \omega) = \underline{\underline{\nabla}}^{\alpha+}_{/\beta}\ \underline{\underline{\Lambda}}_{\beta}(\omega)\ \underline{\underline{D}}_{\beta}(\omega)\ \underline{\underline{\nabla}}^{\beta+}_{\beta/} \ , \tag{77}$$

where

$$
\begin{aligned}
\underset{\approx}{\nabla}^{\alpha+} &= \underset{\approx}{\nabla}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}) = \left(U_j^{\alpha+}(x_j - x_j')\exp(-i\omega\tau_{jj})\delta_{ji}\right) \quad ; \\
\underset{\approx}{\nabla}^{\alpha+}_{/\beta} &= \underset{\approx}{\nabla}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}_{/\beta}) = \left(U_j^{\alpha+}(x_j - x_{\beta j})\exp(-i\omega\tau_{/\beta j})\delta_{ji}\right) \quad ; \\
\underset{\approx}{\nabla}^{\alpha+}_{\alpha/} &= \underset{\approx}{\nabla}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}_{\alpha/}) = \left(U_j^{\alpha+}(x_{\alpha j} - x_j')\exp(-i\omega\tau_{\alpha/j})\delta_{ji}\right) \quad ; \\
\underset{\approx}{\nabla}^{\alpha+}_{\alpha\beta} &= \underset{\approx}{\nabla}^{\alpha+}(\omega\underset{\sim}{\tau}^{\alpha}_{\alpha\beta}) = \left(U_j^{\alpha+}(x_{\alpha j} - x_{\beta j})\exp(-i\omega\tau_{\alpha\beta j})\delta_{ji}\right) \quad .
\end{aligned}
\tag{78}
$$

[cf. equation (40).] Equations (56), (58), (67), and (75) through (77) are identical, except for minor notational differences, with the equations derived, in References 2 and 4, for the impulse response matrix operators and/or functions. In these references, however, the one-dimensionality of the basic dynamic systems was assumed a priori. It may be pointed out that in the frequency domain, temporally stationary models of complexes composed of one-dimensional dynamic systems can be treated with relative ease, even if these dynamic systems are not basic. Thus, for example $C_j^{\alpha}$, in equation (74), may be assumed to be frequency dependent, rendering, in turn, $\underset{\sim}{\tau}^{\alpha}$ frequency dependent. Also, more complex propagation, such as may occur in a model that befits cylindrical geometry, may be treated in the frequency domain more readily than in the temporal domain [4]. However, in these instances one needs to return to the more general equations; the equations for the simple model cannot quite handle such situations. Nonetheless, the formalism herein stands ready, with some increase in cumbersomeness, to accommodate even these deviant situations. This increase in cumbersomeness is miniscule compared with that introduced by increase in dimensionality. As is clearly demonstrated in the wave approach just discussed: increase the dimensionality and the task is increased by much more than a linear proportionality. This leads to a few final remarks. The modal formalism appears less dependent on the spatial dimensionality of the dynamic systems; see Section IV. Of course this is misleading. The spatial dependence of the formalism is removed and stored away in terms of sets of eigenfunctions. However, the eigenfunctions are strongly influenced by the spatial dimensionality of the dynamic systems. Indeed, making the eigenfunctions for all the dynamic systems available for storage is

the greater part of the solution to the spatial dependence of the response of these dynamic systems. The availability and the storage of the eigenfunctions introduce severe limitations on the versatility of the modal formalism. An eigenfunction is usually *fixed in the format* of "standing waves" which gives it quasi-symmetric properties with respect to the spatial dependence. If the response does not follow spatial patterns that can be described by a few of the eigenfunctions, the cumbersomeness of the description may become forbidding. Indeed, for a lopsided response, such as may be caused by excessive damping, significant disparity in the junctions, substantial rectification in the propagation in some of the dynamic systems, etc., the modal formalism may be overtaxed. For example, try and describe the direct part of the response with the modal formalism! On the other hand, the wave formalism does not suffer this "locked-in" limitation; e.g., the direct part of the response is readily determined! However, the spatial dependence in the wave formalism is built piece-by-piece. Therefore, the formalism is cluttered with the spatial dependences. A reduction in the spatial dimensionality of the dynamic systems is communicated directly and dramatically to the wave formalism. Thus, although the two formalisms are formally complimentary, they may also be supplementary. That is, in addition to furnishing the same information, they may also reveal different, yet useful, information in trying to resolve a problem in which the issue is the response of an externally driven complex.

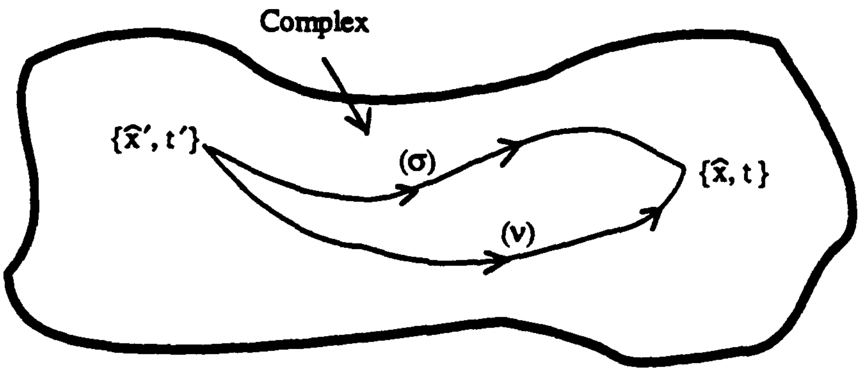


Fig. 1. Schematics of a complex on which are shown two typical paths; path ν and path σ, from the localized external drive positions at $\{\hat{x}', t'\}$ to the localized observation position at $\{\hat{x}, t\}$.

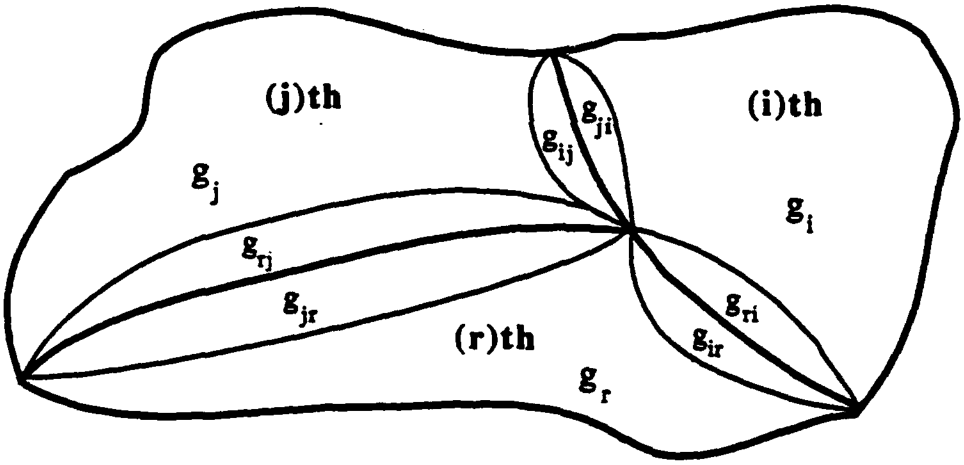


Fig. 2. Schematics of a complex subdivided into (three) dynamic systems. The $g_j$ is the self-impulse response function of the (j)th dynamic system and $g_{ji}$ is the coupling impulse response function between the (i) and the (j)th dynamic systems.

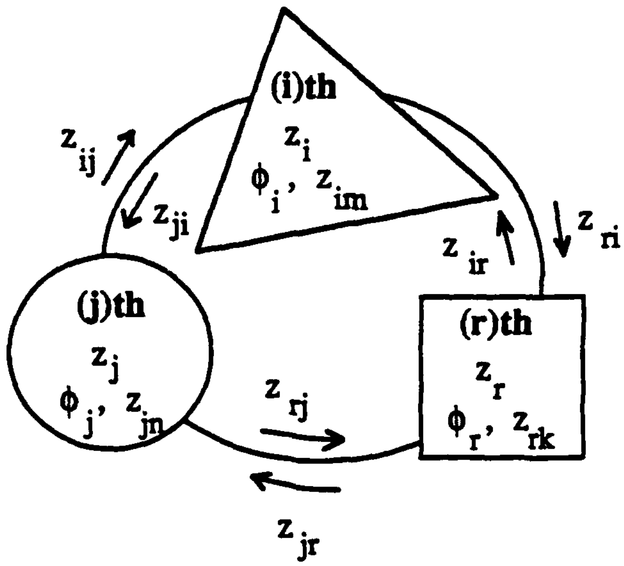


Fig. 3. Schematics of a complex modeled by coupled (three) dynamic systems. The $z_j$ is the self impedance operator of the (j)th dynamic system and $z_{ji}$ is the coupling impedance operator between the (i)th and the (j)th dynamic systems. The self impedance operator $z_j$ defines a set of eigenfunctions $\phi_{jn}$ and a set of corresponding eigen-values $z_{jn}$.

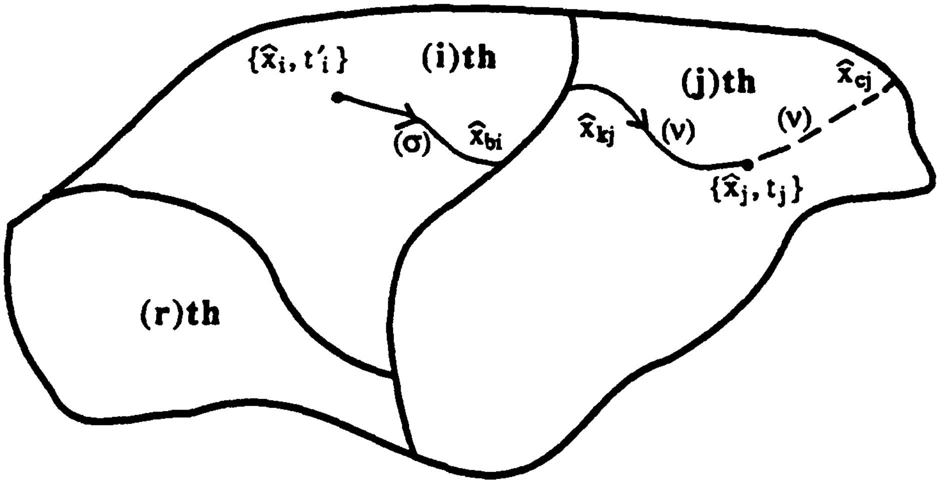


Fig. 4. Schematics of a complex subdivided into (three) dynamic systems. Propagation is initiated in path σ at the position $\hat{x}_i$ in the (i)th dynamic system. The propagation via this path reaches the boundary at position $\hat{x}_{bi}$. A component of the propagation enter the (j)th dynamic system at the position $\hat{x}_{kj}$ and propagates in path n to the observation position $\hat{x}_j$ in the (j)th dynamic system. [The latter propagation may continue until it reaches and is incident on the position $\hat{x}_{cj}$.]

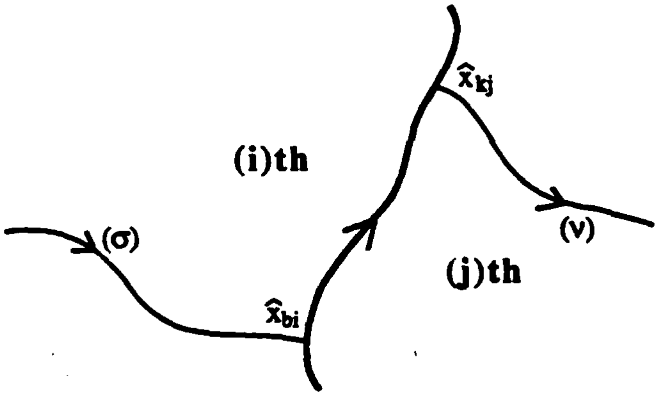


Fig. 4b. Incident position $\hat{x}_{bi}$ and exit position $\hat{x}_{kj}$ are not coincident.

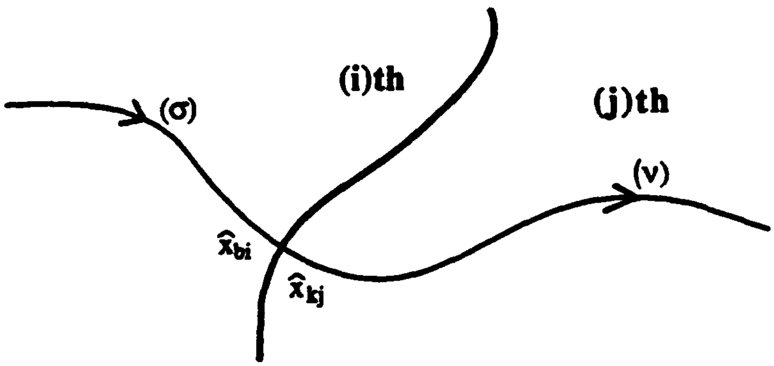


Fig. 4c. Incident position $\hat{x}_{bi}$ and exit position $\hat{x}_{kj}$ are coincident.

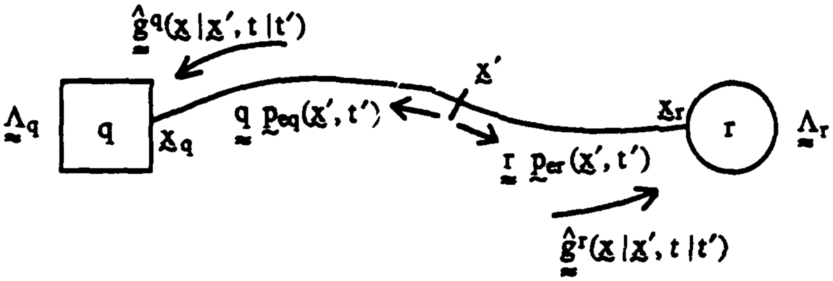


Fig. 5. Schematics of a complex modeled in terms of multiple one-dimensional dynamic systems. Two junctions are necessary; junction r and q. These junctions are defined by the terminal positions $x_\alpha$ and by the junction matrices $\Lambda_\alpha$; $\alpha = r$ and $\alpha = q$.

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